Intransitive indifference with direction-dependent sensitivity*

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Abstract

In much of the literature on intransitive indifference, it is more likely to occur when alternatives have conflicting evaluations than when one alternative dominates the other. To examine such a behavioral pattern, we axiomatize the essentially unique *expected utility with direction-dependent sensitivity* (EUDS) representation on the set of lotteries, which extends the classic models of imperfect discrimination (e.g., Fishburn, 1970a; Luce, 1956) by enabling a direction-dependent just-noticeable difference function. The key axioms for this characterization are *irresolute independence*, wherein mixing alternatives with another alternative may change a strict preference to indifference while preserving indifference, and *strict preference convexity*, which derives the convexity of strict upper and lower contour sets. EUDS can also distinguish two classes of intransitive indifference, that is, those caused by *imperfect discrimination* and *uncertainty about tastes*, the latter of which could not be specified in previous studies on intransitive indifference. We also obtain two special cases of our model, that is, one-directional and categorical sensitivity, which highlight the two classes of intransitive indifference.

Keywords: intransitive indifference, direction-dependence, incomplete preference, transitive core

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1 Introduction

1.1 Background and outline

The theory of choice usually assumes the *transitivity* of the indifference relation, in addition to that of strict preference. However, many authors have argued that the assumption of transitive indifference may be *systematically violated* because of imperfect discrimination between alternatives (e.g., Armstrong, 1939; Fishburn, 1970a, 1970b; Luce, 1956), conflicting evaluations (e.g., Armstrong, 1948, 1950; Fishburn, 1968, 1970b), and justifiable preferences (Lehrer and Teper, 2011), which motivate the analysis of *intransitive indifference*.

One of the most prevalent approaches in this context is *semiorder* (Luce, 1956) and *interval order* (Fishburn, 1970a) models, which describe imperfect discrimination (i.e., insensitivity to small utility differences) using the following representation of a strict preference $\succ$ on the set X of prizes:

$$x \succ y \Longleftrightarrow u(x) > u(y) + \delta(y) \tag{1}$$

for all x, $y \in X$, where u denotes the *utility function* such that $x \succ y$ implies that $u(x) > u(y)$, but not necessarily the converse, and δ denotes the *just-noticeable difference (JND) function*, which generally takes a *finite* value that determines the threshold of the utility difference between alternatives, below which the alternatives cannot be discriminated. Because of its tractability, the latter approach has been applied to various subjects in economic theory, such as choice under risk (e.g., Dalkiran et al., 2018; Gilboa and Lapson, 1995; Vincke, 1980), similarity (e.g., Rubinstein, 1988), and the critical cost-efficiency index (e.g., Dziewulski, 2020).

As (1) indicates, the semiorder/interval order approach determines the JND function δ only using the benchmark (or inferior) alternative y and thus, *disregards* the other (or superior) alternative x. However, in many studies, particularly, those associating intransitive indifference with conflicting evaluations, researchers have hinted at *direction-dependent* sensitivity to a utility difference, that is, δ depending not only on y but also on x. In particular, a dominating alternative is often strictly preferred to a dominated alternative, even if the alternatives differ only by a small margin, whereas alternatives are more likely to be indifferent if neither of them dominates the other (Armstrong, 1948, 1950; Fishburn, 1968, 1970b); the motivating examples in the next subsection expand on this issue. Arguably, such a behavioral pattern can*not* be explained by the existing approach, which assumes a direction-independent JND function.

The primary objective of this study is to explain the latter behavioral implication, by extending the semiorder/interval order approach in the following three respects: first, we assume the set $\mathcal{P}(X)$ of *lotteries* (i.e., the Borel probability measures on X), instead of the set of prizes, as the domain of choice. Second, we axiomatize an essentially unique preference representation in this lottery domain, which we refer to as *expected utility with direction-dependent sensitivity* (EUDS),

$$p \succ q \Longleftrightarrow \int_X u(x)dp > \int_X u(x)dq + \delta(p, q)$$

for all p, $q \in \mathcal{P}(X)$ (Theorems 1 and 2). It generalizes the standard expected utility with the utility function u using the direction-dependent JND function δ; that is, $\delta(p, q)$ typically depends *both* on alternatives p and q, whereas it is constant for any simultaneous shift of p and q. Finally, unlike in previous studies on intransitive indifference, we allow for an *infinite* value of δ in EUDS, in addition to finite values.

Two key axioms exist for this characterization. First, *irresolute independence* is equivalent to the standard independence axiom, except it allows a strict preference to become indifference when the alternatives are mixed with another alternative. Such an axiom establishes a direction-dependent JND function by determining a strict preference or indifference in each direction of comparison between the alternatives. Second, *strict preference convexity* renders the mixture of superior alternatives strictly preferred to the mixture of inferior alternatives and thus, defines convex strict upper and lower contour sets.

Next, readers may suspect that intransitive indifference characterized by EUDS is simply a relabeling of *indecisiveness* (i.e., inability to rank alternatives) in recent models of incomplete preference (e.g., Bewley, 1986; Dubra et al., 2004; Ok et al., 2012) because intransitive indifference and indecisiveness relations are both generally intransitive.[1] However, such a conjecture is inaccurate because intransitive

[1] A similar view is offered in the literature. For example, Kreps (2013, p. 20) concluded that

indifference and indecisiveness may have different welfare implications. By this line of reasoning, we first link EUDS to an incomplete preference model, referred to as *expected multi-utility* (EMU) (Dubra et al., 2004), by considering the transitive core (Nishimura, 2018), that is, the welfare-relevant ranking derived from EUDS (Theorem 3). However, we can only establish the link between indifference in EUDS and indecisiveness in EMU for an *infinite* value of the JND function δ, whereas no such connection exists for finite values (Theorem 4).

These results indicate that EUDS can distinguish two classes of intransitive indifference: the first class is associated with a finite value of the JND function and thus, can be interpreted as *imperfect discrimination* because it establishes finite insensitivity to utility differences. It can also be eliminated by the transitive core; that is, given that $\delta(p,q) < +\infty$, alternative p may be strictly ranked above alternative q by the transitive core, even if p and q are indifferent in EUDS. By contrast, the second class of intransitive indifference is associated with an infinite value of the JND function and can be interpreted as *uncertainty about tastes*, that is, conflicting ex post evaluations. It can survive in the transitive core, that is, if $\delta(p,q) = +\infty$, the indifference in EUDS can be identified with the indecisive transitive core ranking. Although existing approaches, such as semiorder and interval order, focus on the former class of intransitive indifference, the latter class is also essential from the economic point of view because uncertainty plays a crucial role in individual and collective decision-making. Accordingly, one of our contributions is that we establish a general framework that can accommodate the two classes of intransitive indifference and isolate the latter class from the former.

Finally, we focus on the following two special cases of EUDS to clarify the dichotomy of intransitive indifference mentioned above: the first is *one-directional sensitivity*, which obtains a constant JND function δ and the counterpart of the semiorder/interval order approach (Theorem 5). The second special case is *categorical sensitivity*, that is, the JND function δ only takes the value of zero or infinity, which only obtains a strict preference or indifference, depending on the di-

the indifference relation may be intransitive if it is identified with indecisiveness. By contrast, Galaabaatar and Karni (2013) indicated a potential problem caused by defining the indifference relation using the absence of strict preference, in the context of incomplete preference.

rection of comparison (Theorem 6). The latter case is the counterpart of Lehrer and Teper's (2011) justifiability model in our framework, which is another source of intransitive indifference, and thus, the connection between this special case and the corresponding incomplete preference parallels that between the justifiability and Bewley's (1986) Knightian uncertainty models (Corollary 2).

The remainder of this paper is organized as follows: in the next subsection, we present motivating examples. We propose the basic framework and axioms that we focus on in this study in Section 2. In Section 3, we state the main representation theorem and the uniqueness result. We relate EUDS to its counterpart, that is, EMU, in Section 4 and discuss special cases of EUDS in Section 5. We review related literature in Section 6 and make concluding remarks in Section 7.

1.2 Motivating examples

In this section, we examine the following two examples, both of which are motivated by the literature and highlight the main focus of this study.

Example 1 (Choice over multiattribute goods) Consider the following model of choice over multiattribute goods.[2] Let $X = X_1 \times X_2$ be the set of goods, where X_1, $X_2 \subseteq \Re$ are sets of attribute values (e.g., gas mileage and luxury in the case of an automobile). Assume that for $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$, $y = (y_1, y_2) \in X$, good x dominates x' by a small margin (i.e., $x_1 = x'_1 + \epsilon_1$ and $x_2 = x'_2 + \epsilon_2$ for sufficiently small ϵ_1 and ϵ_2) and neither x and y nor x' and y dominate the other (i.e., $x_1 > y_1$, $x_2 < y_2$, $x'_1 > y_1$, and $x'_2 < y_2$, without loss of generality). The decision maker (DM) may readily exhibit a strict preference for good x over x' because the DM recognizes that the former alternative dominates the latter; however, the DM may still be indifferent both between x and y and between x' and y because the DM finds it difficult to discriminate between the alternatives as a result of conflicting attribute values.

[2]This example is motivated by several studies, including those by Armstrong (1948), Fishburn (1970b), May (1954), and Tversky (1969), which associate intransitive preference or intransitive indifference with conflicting attribute values. A similar discussion can be found in the paper by Tversky and Shafir (1992), in the context of choice deferral.

Example 2 (Choice over lotteries) Consider the following model of choice over lotteries.[3] Let $X \subset \Re$ be the set of prizes and $\mathcal{P}(X)$ be the set of lotteries (Borel probability measures on X), endowed with the standard mixture operation. For some p, q, $r \in \mathcal{P}(X)$ and $x \in X$, assume that $p(x+1) = 1$, $q(x) = 1$, $r(x-50) = .5$, and $r(x+100) = .5$. The DM may exhibit a strict preference for p over q because p first-order stochastically dominates q, despite a small difference (equal to one) between the resulting prizes; however, the DM may still be indifferent both between p and r and between q and r, which establishes intransitive indifference because there are no apparent criteria to discriminate between alternatives in each pair.

There are two crucial observations about the examples above: first, the DM's inability to discriminate between alternatives generally depends on the *direction* of comparison between alternatives. More specifically, indifference would seem less likely to occur for choice between dominating and dominated alternatives (in an appropriate sense), which reduces the value of the JND function. By contrast, indifference would be more likely for choice between mutually undominated alternatives, which increases the value of the JND function.[4] Although this view conforms with the findings in many theoretical and experimental studies (Armstrong, 1939, 1948, 1950; Fishburn, 1968, 1970b; May, 1954; Tversky, 1969; Tversky and Shafir, 1992) and is reminiscent of incomplete preferences (Bewley, 1986; Dubra et al., 2004; Ok et al., 2012), the choice patterns described in the examples can*not* be explained by the semiorder and interval order models because the latter approach assigns a fixed value to the JND function for a given benchmark alternative.

Second, the strict upper and strict lower contour sets in these examples can naturally be assumed to be *convex*, as in classical demand theory. Suppose in the multiattribute good choice example that both alternatives z and z' dominate w.

[3] This example was initially considered by Fishburn (1968, 1970b), who was motivated by Keynes (1929). First-order stochastic dominance is also used as a criterion of the "unambiguously better" probability distribution in the literature on risky choice (for an introductory discussion, see, e.g., Mas-Colell et al., 1995).

[4] Indeed, May (1954) suggested that "... it [his experiment] does not prove that individual patterns are always intransitive. It does, however, suggest that where choice depends on conflicting criteria, preference patterns *may* be intransitive unless one criterion dominates" (p. 7).

Then, the DM would readily exhibit a strict preference for a mixture of z and z' over w because the former alternative also dominates the latter. Similarly, if both p and p' first-order stochastically dominate q, in the second example, a mixture of p and p' would also first-order stochastically dominate q, which establishes a strict preference for the mixture over q. Accordingly, both examples are consistent with convex strict upper and strict lower contour sets.

2 Preliminaries and axioms

Let X be a compact metric space of prizes. The set of lotteries (Borel probability measures) on X endowed with the weak convergence topology is denoted by $\mathcal{P}(X)$, and the set of all continuous functions on X endowed with the weak topology, which is equivalent to the sup-norm topology because X is compact, is denoted by $C(X)$. Moreover, let $\mathrm{ca}(X)$ be the set of all Borel signed measures on X, endowed with the weak*-topology,[5] which generates the standard weak convergence topology when it is restricted to $\mathcal{P}(X)$.[6]

We refer to generic elements p, q, r, $\ldots \in \mathcal{P}(X)$ as *alternatives* and assume a binary relation $\succ$ over $\mathcal{P}(X)$ as a primitive,[7] which we refer to as the *strict preference*, that is, $p \succ q$ implies that alternative p is *definitely* preferred to q. We also define the indifference relation $\sim$ as the *absence* of a strict preference; that is, for all p, $q \in \mathcal{P}(X)$, $p \sim q$ if neither $p \succ q$ nor $q \succ p$; that is, $p \sim q$ implies that alternatives p and q cannot be discriminated, rather than being found equally desirable. We define the weak preference $\succsim$ so that for all p, $q \in \mathcal{P}(X)$, $p \succsim q$ if $p \succ q$ or $p \sim q$, and thus, $\succsim$ is complete. In the following analysis, we use the symbols $\succsim$ and $(\succ, \sim)$ interchangeably. Next, for all p, $q \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, $\alpha p + (1-\alpha)q$ denotes the mixture of p and q with probability α, that is, a randomization between

[5] Because X is compact, $\mathrm{ca}(X)$ (endowed with the total variation norm) is isometrically isomorphic to the topological dual of $C(X)$ (endowed with the sup norm).

[6] Note that $\mathrm{ca}(X) = \mathit{span}(\mathcal{P}(X))$.

[7] Instead of the set $\mathcal{P}(X)$ of lotteries, we can set a linear space such as Euclidean space as the domain of choice. Although the latter domain is particularly intuitive in choice over multiattribute goods, we choose $\mathcal{P}(X)$ as the domain of choice because it establishes a comparison with existing lottery choice models, particularly that between our key axioms and the standard independence axiom.

probability distributions generated by p and q. Finally, we define the strict upper and lower contour sets $U_p(\succ)$ and $L_p(\succ)$, given $p \in \mathcal{P}(X)$, respectively, as $U_p(\succ) \equiv \{q \in \mathcal{P}(X) : q \succ p\}$ and $L_p(\succ) \equiv \{q \in \mathcal{P}(X) : p \succ q\}$.

We impose the following axioms on the pair of binary relations $(\succ, \sim)$.

Axiom 1 (Strict partial order) $\succ$ is irreflexive and transitive.

This axiom states that $\succ$ is a *strict partial order* (Suppes, 1957), which is more general than the major intransitive indifference models, such as semiorder (Luce, 1956) and interval order (Fishburn, 1970a). Intuitively, strict preference $\succ$ is considered transitive because it implies the *definite* preference of the DM, which can reasonably be assumed to be fully rational. However, indifference $\sim$ may *not* be transitive because it is defined by the absence of definite (i.e., strict) preference, which seems less consistent than indifference defined by equal desirability. Accordingly, the DM may still strictly prefer, for example, alternative p to r even when the DM is indifferent between p and q, and between q and r. Axiom 1 also implies that $\sim$ is reflexive, that is, $p \sim p$ for all $p \in \mathcal{P}(X)$.

Our second axiom is a weak form of continuity, which is stated with respect to the strict preference $\succ$.

Axiom 2 (Weak continuity) For all p, q, r, $s \in \mathcal{P}(X)$, $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ \alpha r + (1-\alpha)s\}$ is open in $[0,1]$.

The continuity axiom assumed in this paper is weaker than that used by Dubra et al. (2004), whereas the proofs of our main theorems apply their argument.[8] This is because our utility function u only carries a *one-directional* implication (i.e., for all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies that $\int_X u(x)dp > \int_X u(x)dq$, but $\int_X u(x)dp > \int_X u(x)dq$ does *not* necessarily imply that $p \succ q$), and its existence can be proven without assuming the stronger continuity axiom.

[8]Their continuity axiom states that, for all sequences of lotteries $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$, $p_n \succsim q_n$ for all n implies that $\lim_{n\to\infty} p_n \succsim \lim_{n\to\infty} q_n$.

Next, to derive an expected utility representation, we need certain forms of independence, whereas the standard independence axiom (i.e., $p \succ q$ implies that $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$ for all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in [0,1]$) may be too strong to accommodate intransitive indifference (Fishburn, 1968). Accordingly, we consider the following three weakenings of independence in this study:[9] the first weakening allows a strict preference to be altered into indifference after mixing alternatives with another alternative, whereas it preserves indifference.

Axiom 3 (Irresolute independence) For all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in (0,1)$, the following statements hold:

(a) $p \succ q$ implies $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$;

(b) $p \sim q$ implies $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$.

Axiom 3 is equivalent to the standard independence axiom, except that we only allow for a *weak* preference $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$, rather than a strict preference, in the latter half of condition (a). To explain its implication, suppose that alternative p is strictly preferred to q, and let $p' \equiv \alpha p + (1-\alpha)r$ and $q' \equiv \alpha q + (1-\alpha)r$ for some $r \in \mathcal{P}(X)$ and $\alpha \in [0,1]$. The first case covered by Axiom 3(a) retains a strict preference for alternative p' over q', which implies that the DM is entirely definite about the preference for alternative p over q, even after mixing both alternatives with r. However, the second case allows for *indifference* between p' and q' because mixing alternatives may obscure the strict preference, which is the main reason that a thick indifference curve is established. By contrast, condition (b) preserves indifference between p' and q' once indifference between p and q has occurred because the alternatives remain indistinguishable whether they are mixed with another alternative r or not. Fishburn (1968) also suggested Axiom 3(a) as an independence axiom compatible with intransitive indifference, although he provided no specific axiomatization (i.e., the "if-and-only-if" conditions) of a

[9] In this study, we relax independence with respect to when a strict preference should hold, while retaining the linearity of the preference to obtain an expected utility form. Such an assumption typically excludes Allais paradox-type behavior.

preference representation by using it.[10]

Note that by the definition of p' and q' above, $p' - q' = \alpha(p - q)$; that is, mixing alternatives p and q with another alternative reduces the distance between the alternatives (recall that $\alpha \in [0, 1]$), whereas signed measure $p' - q'$ parallels $p - q$. Accordingly, Axiom 3 permits a change from a strict preference to indifference as the two alternatives approach each other, given a fixed direction of comparison between the alternatives, and thus offers a behavioral implication for direction-dependent sensitivity.

The second weakening of independence elaborates a structure of strict preference.

Axiom 4 (Strict preference convexity) For all p, q, p', $q' \in \mathcal{P}(X)$ and $\alpha \in [0, 1]$, $p \succ q$ and $p' \succ q'$ imply $\alpha p + (1 - \alpha)p' \succ \alpha q + (1 - \alpha)q'$.

Axiom 4 renders the mixture of strictly preferred alternatives (i.e., p and p') strictly preferred to the mixture of strictly less preferred alternatives (i.e., q and q'), which implies that hedging between the former alternatives is strictly more valuable than that between the latter. Letting $q = q'$ in Axiom 4 establishes the convexity of strict upper contour sets $U_q(\succ)$ for all $q \in \mathcal{P}(X)$, which conforms with the conclusions of motivating examples in the Introduction and carries an implication similar to that of convex preferences in classical demand theory.[11] Likewise, letting $p = q'$ obtains the convexity of strict lower contour sets $L_p(\succ)$ for all $p \in \mathcal{P}(X)$.

Although Axiom 4 is implied by the standard independence axiom, we still need it to be imposed because our irresolute independence axiom only guarantees a *weak* preference between the mixtures; that is, we may have $\alpha p + (1 - \alpha)p' \sim \alpha q + (1 - \alpha)q'$, even if $p \succ q$ and $p' \succ q'$, and thus, we must explicitly assume the strict

[10] More precisely, the corresponding axiom in Fishburn (1968) states that, for all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in [0, 1]$, $p \succ q$ implies that not $\alpha q + (1 - \alpha)r \succ \alpha p + (1 - \alpha)r$. Under completeness, it is equivalent to Axiom 3(a).

[11] For an introductory explanation of convex preferences in classical demand theory, see, for example, Mas-Colell et al. (1995). However, we should note that Axiom 4 establishes the convexity of *strict* preference to remove the impact of thick indifference curves, whereas classical demand theory usually assumes the convexity of *weak* preference, the latter of which can be translated into our framework that, for all p, q, $r \in \mathcal{P}(X)$, $p \succsim r$ and $q \succsim r$ entail that $\alpha p + (1 - \alpha)q \succsim r$ for all $\alpha \in (0, 1)$.

preference in this case to obtain the convexity of strict upper and strict lower contour sets. Through a motivation similar to ours, Fishburn (1968) and Nakamura (1988) also considered Axiom 4 as weakening independence to characterize intransitive indifference. However, they focused on a direction-*in*dependent JND function by assuming a semiorder or interval order as a primitive, whereas we axiomatize a direction-dependent JND function by starting with a strict partial order.

The third (and final) weakening of independence adds a structure to the indifference relation.

Axiom 5 (Balanced indifference) For all p, q, p', $q' \in \mathcal{P}(X)$, if $p \sim q$ and $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$, then $p' \sim q'$.

The intuition behind this axiom is as follows: suppose that the DM is unable to discriminate between p and q, that is, $p \sim q$. Then, $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$ implies that the change from p to q can be precisely compensated for by the change from p' to q'. Axiom 5 states that for such p, q, p', and q', the DM should naturally be unable to discriminate between p' and q', which establishes $p' \sim q'$.

Note that $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$ is equivalent to $p - q = p' - q'$ (the latter equation is between the signed measures, that is, $p - q$, $p' - q' \in \mathrm{ca}(X)$). Accordingly, Axiom 5 renders the indifference relation invariable with respect to a simultaneous shift of alternatives; that is, the indifference relation depends only on the *direction* of the signed measure generated by alternatives, that is, $p \sim q$ and $p - q = p' - q'$ imply that $p' \sim q'$. The latter property establishes indifference curves *symmetric* with respect to a given alternative q (i.e., $p \sim q$ and $p - q = q - r$ imply that $q \sim r$, for all p, q, $r \in \mathcal{P}(X)$), which also derives the symmetry between strict upper $U_q(\succ)$ and strict lower $L_q(\succ)$ contour sets.

Axiom 5 is implied by the standard independence axiom and similar to those assumed in the literature that derive an additively separable preference representation (e.g., Hyogo, 2007; Karni, 2004). However, our approach differs from those in the following two respects: first, we use Axiom 5 to characterize the indifference relation (and thus, the JND function) rather than obtaining the additive separability of a preference representation as in the literature. Second, unlike the axioms in the literature, we require *equivalence*, not indifference, between $\frac{1}{2}p + \frac{1}{2}q'$ and $\frac{1}{2}q + \frac{1}{2}p'$ because our model generally admits thick indifference curves, and thus, we need to accurately measure the compensation effect on the indifference relation.

Finally, we assume the standard nontriviality axiom, that is, there exists at least one alternative pair, one of which is strictly preferred to the other. It is required for identifying the utility and JND functions simultaneously.[12]

Axiom 6 (Nontriviality) There exist p, $q \in \mathcal{P}(X)$ such that $p \succ q$.

3 Main theorems

In this section, we state our main theorems. First, we define two components of our preference representation: the *utility function* and *JND function*.

A utility function partially represents the strict preference as follows:

Definition 1 (Utility function) For the strict preference $\succ$, a function $u \in C(X)$ is referred to as the *utility function of* $\succ$ if for all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies that $\int_X u(x)dp > \int_X u(x)dq$.

Definition 1 describes the standard utility function over lotteries except for its one-directional implication; that is, we do *not* assume that $\int_X u(x)dp > \int_X u(x)dq$ implies $p \succ q$, unlike a utility function that represents a weak order.

Next, a JND function denotes the threshold of the utility difference between alternatives below which the alternatives cannot be discriminated. Let $H(u) \equiv \{(p, q) \in \mathcal{P}(X) \times \mathcal{P}(X) : \int_X u(x)dp > \int_X u(x)dq\}$, that is, $H(u)$ is the set of alternative pairs (p, q), wherein p establishes a higher expected utility than q, given a utility function u.

Definition 2 (JND function) For a utility function $u \in C(X)$, a function $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \Re_+$ is referred to as the *JND function given* u if it satisfies the

[12] If all alternatives are indifferent, the functional form of EUDS, which we explain in the next section, cannot determine whether the utility function is constant or the JND function is infinite.

following conditions:

(a) $\delta(p,q) = \delta(p',q')$ for all p, q, p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p - q = \lambda(p' - q')$;
(b) $\delta(\alpha p + (1-\alpha)p', q) \leq \alpha\delta(p,q) + (1-\alpha)\delta(p',q)$ for all p, p', $q \in \mathcal{P}(X)$ and $\alpha \in [0,1]$;
(c) $\delta(p,q) = +\infty$, if $(p,q) \notin H(u)$ or $\delta(p,q) \geq \sup_{p,q\in\mathcal{P}(X)} \left(\int_X u(x)dp - \int_X u(x)dq\right)$.

Condition (a) renders the JND function δ *direction-dependent*, that is, for all p, q, p', $q' \in \mathcal{P}(X)$, $\delta(p,q) = \delta(p',q')$ whenever the signed measure $p-q$ parallels $p'-q'$. Next, condition (b) implies that for an arbitrary $q \in \mathcal{P}(X)$, $\delta(\cdot, q)$ is concave, and it establishes the convexity of strict upper and strict lower contour sets, together with Definition 3. Finally, condition (c) states that δ takes an infinite value for alternative pairs (p,q) if p's expected utility is equal to or lower than q's, or δ exceeds the supremum of the possible utility difference, which establishes a well-defined JND function.

Now, integrating the ideas explained above, we present our preference representation.

Definition 3 (Expected utility with direction-dependent sensitivity) For a utility function u and JND function δ, the function pair (u,δ) is referred to as the EUDS representation of $(\succ, \sim)$ if it satisfies the following conditions:

(a) for all p, $q \in \mathcal{P}(X)$, $p \succ q$ if and only if $\int_X u(x)dp > \int_X u(x)dq + \delta(p,q)$;
(b) for all p, $q \in \mathcal{P}(X)$, $p \sim q$ if and only if $\int_X u(x)dq - \delta(q,p) \leq \int_X u(x)dp \leq \int_X u(x)dq + \delta(p,q)$.

Definition 3 implies that both weak and strict preferences can be specified by the *combination* of utility and JND functions: condition (a) establishes the strict preference for alternative p over q if and only if the difference in expected utility, $\int_X u(x)dp - \int_X u(x)dq$, is greater than the value $\delta(p,q)$ of the JND function, in which case the DM can successfully discriminate between alternatives p and q. By contrast, condition (b) guarantees indifference between p and q if and only if $\int_X u(x)dp - \int_X u(x)dq$ is within the range of $-\delta(q,p)$ and $\delta(p,q)$, which follows from condition

(a) and the completeness of $\succsim$. Accordingly, EUDS reduces to the standard expected utility if $\delta(p,q) = 0$ for all $(p,q) \in H(u)$.

This definition, together with Definitions 1 and 2, characterizes the *direction-dependence* of EUDS; that is, whether a strict preference or indifference holds between p and q depends not only on the difference of the expected utility between them but also on the signed measure $p - q$. The latter property embodies the class of intransitive indifference that we discussed in the motivating examples in the Introduction. Definition 2(a) also establishes symmetric strict upper $U_q(\succ)$ and strict lower $L_q(\succ)$ contour sets given q, that is, for all p, q, $r \in \mathcal{P}(X)$, such that $p - q = q - r$, $p \in U_q(\succ)$ whenever $r \in L_q(\succ)$.

The next theorem is the main result in this study.

Theorem 1 *The following statements are equivalent.*

(a) Preference pair $(\succ, \sim)$ *satisfies Axioms 1–6.*

(b) Preference pair $(\succ, \sim)$ *admits an EUDS representation* (u, δ).

The proof of the theorem is in the Appendix, and its intuition is as follows: We first obtain the transitivity of the strict preference by assuming a strict partial order. Together with irresolute independence and strict preference convexity, we can also derive a closed convex cone $D(\succ)$ that includes, but is not generally equal to, the strict upper contour set given an alternative (Lemma 1). Accordingly, we obtain a closed convex set $\mathcal{U}$ of utility functions u in $C(X)$, that is, for all $u \in \mathcal{U}$, $p \succ q$ implies $\int_X u(x)dp > \int_X u(x)dq$ and $\int_X u(x)dp \geq \int_X u(x)dq$ implies $p \succsim q$ (Lemma 2), in a manner similar to that of Dubra et al. (2004). Finally, we derive a JND function δ with the help of irresolute independence and strict preference convexity, in addition to other axioms (Lemma 3).

The proof sketch explains why we can successfully obtain a direction-dependent JND function, unlike previous studies that only admitted a direction-*in*dependent JND function in the lottery domain (e.g., Dalkiran et al., 2018; Vincke, 1980): the latter approach typically assumes a semiorder as a primitive, which precludes a direction-dependent JND function; we provide a more detailed discussion in Section 5.1. By contrast, we start with a strict partial order, which is considerably more general than semiorder. The assumption gives our model the flexibility to obtain a direction-dependent JND function, whereas it needs some supplementary axioms, such as irresolute independence and strict preference convexity, to establish a well-structured preference representation. As we indicate later, EUDS can also accommodate a direction-independent JND function as a special case. Accordingly, we develop a framework that can characterize a broad range of intransitive indifference on the set of lotteries, including both direction-dependent and direction-independent cases, by relaxing the semiorder/interval order assumption.

Next, we state our uniqueness result. To accomplish this, we define an operator $\langle\cdot\rangle : 2^{C(X)} \to 2^{C(X)}$ as $\langle\mathcal{V}\rangle \equiv \mathrm{cl}(\mathrm{cone}(\mathcal{V}) + \{\theta\mathbf{1}_X\}_{\theta\in\Re})$ for all $\mathcal{V} \subseteq C(X)$, where $\mathrm{cl}(\mathcal{V})$ denotes the closure of set $\mathcal{V}$ with respect to the weak topology on $C(X)$ and $\mathrm{cone}(\cdot)$ denotes the conic hull of $(\cdot)$, following Dubra et al.'s (2004) approach. The next theorem guarantees the uniqueness of the abovementioned set $\mathcal{U}$ of possible utility functions, under the operator $\langle\cdot\rangle$. We write δ_u to denote the JND function given a $u \in C(X)$, wherein EUDS (u, δ_u) represents $\succsim$.

Theorem 2 *There exists a closed and convex* $\mathcal{U} \subseteq C(X)$*, such that* $(\succ, \sim)$ *admits an EUDS representation* (u, δ_u) *if and only if* $u \in \langle\mathcal{U}\rangle$*. Moreover, for each* $u \in \langle\mathcal{U}\rangle$*,* δ_u *is unique.*

Theorem 2 establishes the essential uniqueness of the set $\mathcal{U}$ of utility functions u, and the corresponding JND function δ_u, that represent the preference. The set $\mathcal{U}$ is generally non-singleton because indifference curves in our model are generally thick and thus, generally admit multiple hyperplanes that separate strict upper and lower contour sets, each of which corresponds to a $u \in \mathcal{U}$.

If there is only one such hyperplane, the set $\mathcal{U}$ becomes a singleton, that is, $\mathcal{U} = \{u\}$ for some $u \in C(X)$. In this case, the uniqueness result is reduced to that of the standard expected utility, that is, uniqueness up to a positive affine transformation. A typical example of this case is a direction-independent JND function, which we axiomatize in Section 5.1. Moreover, we also obtain the standard uniqueness result for a non-singleton $\mathcal{U}$ by restricting our attention to a $u \in \mathcal{U}$ with a specific property, such as the centroid of convex cone $\mathcal{U}$ in the case in which X is Euclidean space.

Now, as anticipated, EUDS can explain the examples considered in the Introduction.

Example 1' (Choice over multiattribute goods, revisited) Assume, for simplicity, an additive separable and linear utility function, that is, there exist some a_1, $a_2 > 0$ such that $u(z_1, z_2) = a_1 z_1 + a_2 z_2$ for all $z = (z_1, z_2) \in X$. For all $v = (v_1, v_2)$, $w = (w_1, w_2) \in X$, which can be considered as degenerate lotteries, let $\delta(v, w)$ be sufficiently small (typically zero) if v dominates w and sufficiently large (typically infinity) if neither v nor w dominates the other. Then, for x, x', $y \in X$ defined in Example 1, we obtain $x \succ x'$, $x \sim y$, and $x' \sim y$. We can also extend δ to $\mathcal{P}(X) \times \mathcal{P}(X)$ so that all the conditions in Definition 2 are satisfied; in particular, $\delta(\cdot, q)$ can naturally be assumed to be a concave function because the set of alternatives that dominate a given alternative is convex.

Example 2' (Choice over lotteries, revisited) Assume that $u(x)$ is increasing with respect to $x \in X$ (recall that $X \subset \Re$) and that for all p', $q' \in \mathcal{P}(X)$, $\delta(p', q')$ is sufficiently small if lottery p' first-order stochastically dominates q' and $\delta(p', q')$ is sufficiently large otherwise. Then, for p, q, $r \in \mathcal{P}(X)$ defined in Example 2, we obtain $p \succ q$, whereas $p \sim r$ and $q \sim r$. Again, this configuration is compatible with all the conditions in Definition 2, particularly the concavity of δ because the set of lotteries that first-order stochastically dominate a given lottery is convex.[13]

Finally, we should note that the JND function δ in EUDS may be discontinuous with respect to each variable, whereas the utility function $u \in C(X)$ is guaranteed to be continuous. For example, let $\hat{D}(\succ) = \{\lambda(p' - q') : p', q' \in \mathcal{P}(X),\ \lambda > 0,\ p' \succ q'\}$,

[13] Assume that $|X| = n$ and $x_1 < \ldots < x_n$. Let F_p and F_q be the cumulative distribution functions generated by lotteries $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ (p_i and q_i for $i = 1, \ldots, n$ denote the probabilities assigned to prize x_i by p and q), respectively. Then, p first-order stochastically dominating q implies that $F_p(x) \leq F_q(x)$ for all $x \in X$, and it follows that $\sum_{i=1}^{j} p_i \leq \sum_{i=1}^{j} q_i$, that is, $\sum_{i=1}^{j}(p_i - q_i) \leq 0$, for all $j = 1, \ldots, n$. Accordingly, $\{p : p$ first-order stochastic dominates $q\}$ is convex because it is the intersection of half-spaces in $\mathrm{ca}(X)$. A similar argument also holds for an infinite X under continuity.

that is, $\hat{D}(\succ)$ is the convex cone generated by alternatives, one of which is strictly preferred to the other.[14] For all p, $q \in \mathcal{P}(X)$, let δ be such that $\delta(p,q) = 0$ if $p - q \in \hat{D}(\succ)$ and $\delta(p,q) = +\infty$ otherwise. Such a δ conforms with Definition 2 and is compatible with all of Axioms 1–6. However, δ is discontinuous with respect to the first variable because letting $p - q \in \partial(\mathrm{cl}(\hat{D}(\succ)))$ (the closure and boundary are taken in the weak*-topology) and $p' - q \in \hat{D}(\succ)$ entails that $\delta(p,q) = +\infty$ and $\delta(\alpha p + (1-\alpha)p', q) = 0$ for all $\alpha \in [0,1)$. Although such a discontinuity may seem to be the result of the weak continuity that we impose, it can*not* be eliminated under a stronger continuity axiom, such as that assumed by Dubra et al. (2004) (i.e., for all $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$, $p_n \succsim q_n$ for all n implies that $\lim_{n\to\infty} p_n \succsim \lim_{n\to\infty} q_n$); that is, the abovementioned example is also consistent with the stronger continuity axiom.

4 Relating EUDS to incomplete preferences

Up to this point, we have axiomatized EUDS, which is a model of intransitive indifference, and obtained its uniqueness. However, in this section, we relate EUDS to an incomplete preference model, that is, EMU, which is significant for the following two reasons.

First, a particular type of intransitive indifference may be relevant to indecisiveness (inability to rank alternatives) in an incomplete preference model: as we indicated in the Introduction, alternatives with conflicting evaluations are more likely to cause intransitive indifference than dominating and dominated alternatives, which suggests that the conflicting evaluations obscure the distinction between the alternatives. However, a similar implication has been also discussed in the context of incomplete preference (Bewley, 1986; Dubra et al., 2004; Ok et al., 2012).

Second, relating EUDS to an incomplete preference may facilitate a welfare evaluation: an intransitive indifference model such as EUDS generally causes a cycle;[15] for example, $x \succ x' \sim y \sim x$ in Example 1' in the previous section. Such a violation of transitivity can be regarded as irrational and complicate a welfare comparison

[14] We also use the set $\hat{D}(\succ)$ in the proof of Theorem 1.

[15] A sequence of alternatives $(p_i)_{i=1}^{n}$ is referred to as a *cycle* if $p_1 \succsim \dots \succsim p_n \succsim p_1$ with at least one strict preference.

of alternatives. However, inferring a transitive (but potentially incomplete) ranking from EUDS may resolve this issue because the obtained ranking can be considered as rational and thus, welfare-relevant.

4.1 Transitive core of EUDS

As a means of relating EUDS to an incomplete preference model, we focus on the *transitive core* (Nishimura, 2018) of preference pair $(\succ, \sim)$ in this study.[16]

An alternative approach to relate EUDS to an incomplete preference might be to define indecisiveness by the *absence* of (weak or strict) preference (e.g., Galaabaatar and Karni, 2013; Kreps, 2013). However, compared with the latter approach, the transitive core has two advantages. First, the transitive core establishes a more solid foundation for the welfare evaluation because it can *necessarily* determine a welfare ranking if the alternatives admit no cycle. By contrast, the alternative approach may regard the alternatives as indecisive in the corresponding incomplete ranking whenever they are indifferent in the initial preference. Such a property is problematic because it may reject a welfare comparison between alternatives that are indifferent but *never* involved in a cycle. Second, and more importantly, the transitive core can successfully distinguish two classes of intransitive indifference in EUDS: one is relevant to an incomplete preference and the other is not. As we indicate shortly, such a distinction is crucial for characterizing the exact relationship between intransitive indifference and indecisiveness. However, we cannot obtain the latter result by simply defining indecisiveness by the absence of (weak or strict) preference because such a definition cannot isolate indecisiveness from indifference in our setting, wherein indifference is already defined by the absence of strict preference.[17]

Now, we define the transitive core of $(\succ, \sim)$ as follows:

[16] The transitive core is also referred to as the *trace* in the theory of semiorders and interval orders. For an extensive survey, see, for example, Bouyssou and Pirlot (2005). A similar idea is also considered in the context of incomplete preference (e.g., Galaabaatar and Karni, 2013).

[17] In our model, defining an indecisiveness relation $\hat{\bowtie}$ by the absence of strict preference (i.e., for all $p, q \in \mathcal{P}(X)$, $p \,\hat{\bowtie}\, q$ if neither $p \succ q$ nor $q \succ p$) establishes the equivalence of $\hat{\bowtie}$ to the intransitive indifference relation $\sim$ and thus, indecisiveness is identical to indifference. Alternatively, if we define an indecisiveness relation $\tilde{\bowtie}$ by the absence of weak preference (i.e., for all $p, q \in \mathcal{P}(X)$, $p \,\tilde{\bowtie}\, q$ if neither $p \succsim q$ nor $q \succsim p$), $\tilde{\bowtie}$ is empty because $\succsim$ is complete.

Definition 4 (Transitive core) For any preference pair $(\succ, \sim)$, we refer to a binary relation $\succsim^{TC}$ as the *transitive core of* $(\succ, \sim)$ if for all p, $q \in \mathcal{P}(X)$,

$$p \succsim^{TC} q \text{ if and only if} \begin{cases} r \succsim p \text{ implies } r \succsim q \\ q \succsim r \text{ implies } p \succsim r \end{cases} \quad \textit{for all } r \in \mathcal{P}(X).$$[18]

Nishimura (2018) indicated that the transitive core is generally transitive, as its name suggests, but possibly incomplete. Accordingly, for all p, $q \in \mathcal{P}(X)$, we write $p \bowtie^{TC} q$ if neither $p \succsim^{TC} q$ nor $q \succsim^{TC} p$, in which case, we say that the transitive core ranking is *indecisive* between alternatives p and q.

As a candidate for the representation of the transitive core of EUDS, we consider the following incomplete preference model, which is referred to as EMU (Dubra et al., 2004).

Definition 5 (Expected multi-utility) Given a set $\mathcal{V} \subseteq C(X)$, we refer to $\succsim_{\mathcal{V}}^{EMU}$ as the *EMU preference with* $\mathcal{V}$, if for all p, $q \in \mathcal{P}(X)$,

$$p \succsim_{\mathcal{V}}^{EMU} q \text{ if and only if } \int_X v(x)dp \geq \int_X v(x)dq \text{ for all } v \in \mathcal{V}.$$

Similarly to the previous case, for all p, $q \in \mathcal{P}(X)$, we write $p \bowtie_{\mathcal{V}}^{EMU} q$ if neither $p \succsim_{\mathcal{V}}^{EMU} q$ nor $q \succsim_{\mathcal{V}}^{EMU} p$.

EMU describes the *unanimity* rule that addresses conflicting evaluations caused by uncertainty about tastes; that is, alternative p is preferred to q only if the utility functions in $\mathcal{V}$ unanimously rank p above q, whereas the ranking between p and q is indecisive if such unanimity is not achieved.

The following theorem links the transitive core of an EUDS preference to EMU.

Theorem 3 *Assume that* $(\succ, \sim)$ *admits an EUDS representation* (u, δ)*. Then,* $\succsim^{TC} = \succsim_{\mathcal{U}}^{EMU}$*, wherein* $\mathcal{U}$ *is the set of utility functions defined in Theorem 2.*

The proof is in the Appendix. Theorem 3 indicates that the transitive core of EUDS is equivalent to EMU with the set $\mathcal{U}$ of utility functions characterized in

[18] Recall that $p \succsim q$ implies $p \succ q$ or $p \sim q$.

Theorem 2; that is, if our welfare evaluation relies on the transitive core as suggested by Nishimura (2018), it can be determined by the unanimity rule of all the possible utility functions in $\mathcal{U}$.

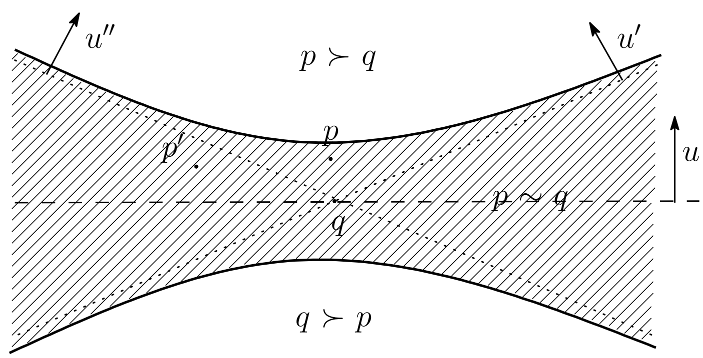

Figure 1: Indifference curve in EUDS

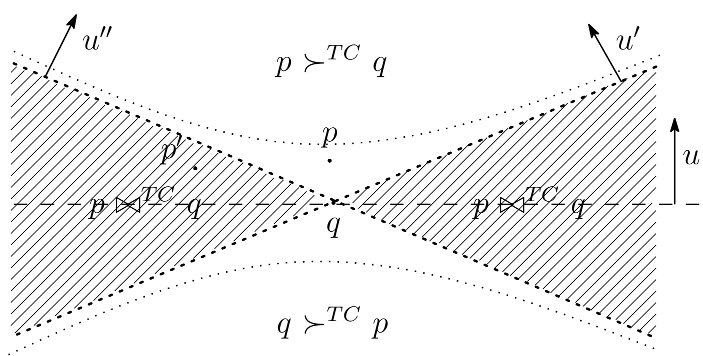

Figure 2: Transitive core of EUDS

Figure 1 illustrates the indifference curve through q in EUDS, depicted in the lottery space $\mathcal{P}(X)$. A typical example of such a graphical representation is the Marschak-Machina triangle for the three-outcome case. The broken line denotes the hyperplane generated by a utility function u, which corresponds to a linear indifference curve in the standard expected utility framework. The shaded area also denotes the (thick) indifference curve through q, generated by u and the JND function, with the bold lines denoting its boundary. Finally, the dotted lines denote the most extreme hyperplanes through q included in the indifference curve, which are generated by normal vectors (i.e., utility functions) u', $u'' \in \mathcal{U}$.

By contrast, the shaded area in Figure 2 depicts the indecisive area given q in the transitive core, which is derived from the thick indifference curve of EUDS in Figure 1. It is defined by the area wedged between the most extreme hyperplanes that we mentioned above (the dotted lines) and thus, attributes the indecisiveness in the transitive core to conflicting utility functions in $\mathcal{U}$. For example, alternative p' in Figure 2 is in the indecisive area; that is, p' is in the upper half-space through q generated by utility function u' and in the lower half-space through q generated by utility function u''. Consequently, p' yields a higher expected utility than q given u' and a lower expected utility given u'', which causes conflicting evaluations. However, the boundary of the indecisive area, that is, the most extreme hyperplanes, is excluded from the indecisive area and thus, it is depicted by the dotted lines rather than the solid lines. In the next section, we discuss such an implication in more detail.

4.2 Separating two classes of intransitive indifference

As we argued in the previous section, intransitive indifference and indecisiveness are often undifferentiated in the literature. However, in this section, we isolate two distinct classes of intransitive indifference in EUDS, that is, imperfect discrimination and uncertainty about tastes, and only associate the *latter* class of intransitive indifference with indecisiveness.

First, we define $I \equiv \{(p,q) : (p,q) \in H(u) \text{ and } \delta_u(p,q) = +\infty \text{ for some } u \in \mathcal{U}\}$, that is, I is the set of nontrivial alternative pairs that generate an infinite value of the JND function and thus, can never be discriminated, given some utility function u. The set I is closed in the product topology on $\mathcal{P}(X) \times \mathcal{P}(X)$.

The following theorem determines exactly when intransitive indifference is relevant to indecisiveness by relating the values of the JND function to a strict preference or indecisiveness in the transitive core.

Theorem 4 *Assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p,q) \in H(u)$ be such that $(p,q) \notin \partial(I)$. Then, the following statements hold:*

(a) $\delta(p,q) < +\infty$ *if and only if* $p \succ^{TC} q$ *(or equivalently,* $p \succ^{EMU}_{\mathcal{U}} q$*);*
(b) $\delta(p,q) = +\infty$ *if and only if* $p \bowtie^{TC} q$ *(or equivalently,* $p \bowtie^{EMU}_{\mathcal{U}} q$*).*

The proof is in the Appendix. To clarify the implications of Theorem 4, suppose that, for some $(p,q) \in H(u)$ such that $(p,q) \notin \partial(I)$, p and q are indifferent. If $\delta(p,q)$ is *finite*, then case (a) associates indifference with *decisiveness* (i.e., strict preference) in the transitive core. The intransitive indifference of this class can be interpreted as *imperfect discrimination* because a finite value of δ denotes an inability to detect a small difference in the expected utility of alternatives. It can also be *eliminated* by the transitive core because it establishes the decisive transitive core ranking; that is, we may have $p \sim q$ and $p \succ^{TC} q$ if $\delta(p,q) < +\infty$. However, if $\delta(p,q)$ is *infinite*, case (b) relates indifference to *indecisiveness* in the transitive core. The intransitive indifference of this class can be attributed to *uncertainty about tastes* because $\succsim^{TC} = \succsim^{EMU}_{\mathcal{U}}$ from Theorem 3, and thus, the indecisiveness in the transitive core is relevant to conflicting evaluations of alternatives. In contrast to the previous case, the latter class of intransitive indifference *cannot* be eliminated by the transitive core; that is, the indifference in EUDS remains as indecisiveness in the transitive core and we necessarily have $p \sim q$ and $p \bowtie^{TC} q$ if $\delta(p,q) = +\infty$. To summarize, Theorem 4 can successfully recover the welfare ranking (i.e., the decisive transitive core ranking) for the first class of intransitive indifference, but *not* necessarily for the second class.

Figures 1 and 2 illustrate the abovementioned implications: Figure 1 indicates that $p \sim q$ and $p' \sim q$ because both p and p' are in the thick indifference curve through q. However, Figure 2 implies that $p \succ^{TC} q$ and $p' \bowtie^{TC} q$ because p is outside the indecisive area given q, whereas p' is inside it. Theorem 4 attributes such a contrast to the difference in the value of the JND function δ: Figure 1 indicates that $\delta(p,q)$ is finite, because p leaves the indifference curve through q as p departs from q, while preserving the direction of signed measure $p - q$. Accordingly, the indifference between alternatives p and q in EUDS can be interpreted as imperfect discrimination and thus, Theorem 4(a) establishes a *strict* transitive core ranking between p and q as in Figure 2. By contrast, we can regard $\delta(p',q)$ as infinite because p' stays within the indifference curve in Figure 1, however much p' departs from q, provided the direction of $p' - q$ is unchanged. Theorem 4(b) associates such indifference with uncertainty about tastes and establishes an *indecisive* transitive core ranking between p' and q as in Figure 2.

Much of the literature on intransitive indifference, including the semiorder/interval order approach, assumes a finite-valued JND function in its preference representations (e.g., Dalkiran et al., 2018; Fishburn, 1970a; Luce, 1956; Vincke, 1980). Theorem 4(a) implies that the literature focuses on the first class of intransitive indifference discussed in the previous paragraph, that is, that caused by imperfect discrimination and a finite value of the JND function. However, as we indicated, a specific class of intransitive indifference is more relevant to conflicting evaluations of alternatives than imperfect discrimination (e.g., Armstrong, 1948; Fishburn, 1970b; May, 1954; Tversky, 1969). Theorem 4(b) associates such a behavioral implication with the second class of intransitive indifference, that is, that caused by uncertainty about tastes and an infinite value of the JND function. Accordingly, one of our contributions is that we establish a general framework that can successfully accommodate (and distinguish between) both classes of intransitive indifference. The key idea is that we allow the JND function to yield an infinite value, in addition to a finite value, which was unattainable in previous studies. The transitive core also plays an essential role in this analysis, and sets our approach apart from the literature that defined indecisiveness by the absence of strict preference.

Finally, we exclude alternative pairs (p, q) on the boundary of I to state Theorem 4. This is because if $(p, q) \in \partial(I)$, $\int_X u(x)dp = \int_X u(x)dq$ for some $u \in \mathcal{U}$ and $\int_X u'(x)dp > \int_X u'(x)dq$ for all $u' \in \mathcal{U}$, with $u' \neq u$. It follows from $\succ^{TC} = \succ^{EMU}_{\mathcal{U}}$ that $p \succ^{TC} q$ for any such alternatives p and q; that is, p is strictly ranked above q by the transitive core, whereas p and q are indifferent in EUDS because $(p, q) \in I$, which may be counterintuitive.

One possible settlement of this boundary issue is to define the following binary relation $\widehat{\succ}^{TC}$ on $\mathcal{P}(X)$: for all p, $q \in \mathcal{P}(X)$, $p \,\widehat{\succ}^{TC}\, q$ if and only if for all r, $s \in \mathcal{P}(X)$, there is $\epsilon > 0$, such that $(1-\alpha)p + \alpha r \succ^{TC} (1-\alpha)q + \alpha s$ for all $\alpha \in [0, \epsilon]$. We can easily indicate that $p \,\widehat{\succ}^{TC}\, q$ if and only if

$$\int_X u(x)dp > \int_X u(x)dq \text{ for all } u \in \mathcal{U}.$$

A similar technique was used by Cerreia-Vioglio et al. (2020) in the context of incomplete preference under uncertainty. The derived ranking $\widehat{\succ}^{TC}$ is the algebraic interior of $\succ^{TC}$, which is equivalent to the interior of $\succ^{TC}$ because $\succ^{TC}$ evidently has a nonempty interior. We also denote by $\widehat{\bowtie}^{TC}$ the indecisiveness relation derived from $\widehat{\succ}^{TC}$. Now, the following corollary connects the induced indecisiveness relation $\widehat{\bowtie}^{TC}$ to an infinite value of δ, including the case in which the alternative pair (p, q) is on the boundary of I. We exclude $p = q$ from the statement below because indifference naturally holds in such a case.

Corollary 1 *Assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p, q) \in H(u)$ be such that $p \neq q$. Then, the following statements hold:*

(a) $\delta(p, q) < +\infty$ if and only if $p \mathrel{\widehat{\succ}^{TC}} q$;
(b) $\delta(p, q) = +\infty$ if and only if $p \mathrel{\widehat{\bowtie}^{TC}} q$.

5 Special cases

In this section, we explore two important special cases of EUDS, each of which embodies a specific class of intransitive indifference discussed in the previous section.

5.1 One-directional sensitivity

First, we extend Luce's (1956) semiorder approach to choice over lotteries, which establishes a constant JND function. To accomplish this, we consider the following axiom, which is the adaptation of Fishburn's (1970a) axiom.

Axiom 7 (Interval order) For all p, q, r, $s \in \mathcal{P}(X)$, $p \succ q$ and $r \succ s$ imply that $p \succ s$ or $r \succ q$.

To interpret Axiom 7, assume the existence of a weak order over alternatives consistent with the strict preference $\succ$, and let $p \succ q$ and $r \succ s$. The first possible case is that p is the "best" and s is the "worst" among all the alternatives, according to the weak order. Then, we can easily anticipate a strict preference for the "best" alternative p over the "worst" alternative s, given that strict preferences are already established between the extreme and intermediate alternatives. Second, suppose that p is the "best" and q is the "worst." In such a case, we can naturally expect a strict preference for the "best" alternative p to the "second-worst" alternative s because a strict preference holds between the "second-best" alternative r and the "second-worst" alternative s. Implications for the other cases can be similarly discussed.

The above discussion indicates that Axiom 7 alludes to the direction-*in*dependence of the strict preference because it establishes a strict preference only by the "distance" between the rankings of alternatives; that is, once a strict preference has held between some alternatives (i.e., $p \succ q$ and $r \succ s$), a strict preference is also established between the alternatives whose rankings are further apart (i.e., p and s in the examples in the previous paragraph). Indeed, Fishburn (1970a) derived from Axiom 7 a direction-independent (but not generally constant) JND function. Conversely, a typical violation of Axiom 7 is that $p \succ q$, $r \succ s$, $p \sim s$, and $r \sim q$ for some alternatives p, q, r, and s, which is reminiscent of a direction-independent preference that EUDS assumes.

Now, we consider the following special case of EUDS.

Definition 6 (Expected utility with one-directional sensitivity) EUDS (u, δ) is referred to as the *expected utility with one-directional sensitivity (EUOS)* (u, k) if there exists $k > 0$ such that $\delta(p, q) = k$ for all $(p, q) \in H(u)$.

Definition 6 establishes a constant value for the JND function, except for the case in which $\int_X u(x)dp \leq \int_X u(x)dq$, which entails that $\delta(p, q) = +\infty$ by definition. The next theorem indicates that this special case can be derived from Axiom 7, along with other axioms.

Theorem 5 *The following statements are equivalent.*

(a) Preference pair $(\succ, \sim)$ satisfies Axioms 1–7.

(b) Preference pair $(\succ, \sim)$ admits an EUOS representation (u, k). Moreover, u in the EUOS representation is unique up to a positive affine transformation (and k is uniquely defined by each u).

The proof is in the Appendix and uses the following intuition: first, together with other axioms, interval order derives the convexity of the indifference curve, that is, for all p_1, p_2, $q \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, $p_1 \sim q$ and $p_2 \sim q$ imply that $\alpha p_1 + (1-\alpha)p_2 \sim q$. It follows from strict preference convexity that the boundary between the indifference curve and the closure of the strict upper contour set, given an alternative, forms a hyperplane. The latter property eliminates the direction-dependence of the preference and establishes a constant JND function.

EUOS is the counterpart of the semiorder model (Dalkiran et al., 2018; Luce, 1956; Vincke, 1980) in our setting. In particular, it is comparable with the models of Dalkiran et al. (2018) and Vincke (1980), who characterized a preference representation over lotteries. However, there are two major differences between their approach and ours. First, our starting point differs from theirs, as we mentioned; that is, they assumed a semiorder, which readily establishes a constant JND function, whereas we first derive EUDS from a *strict partial order*, a class of preferences more general than semiorder, and obtain EUOS as a special case. The readers may wonder why Theorem 5 only additionally needs interval order (i.e., Axiom 7), which generally obtains a *non-constant* JND function, to characterize EUOS. To answer such a question, we should note that establishing a constant JND function generally necessitates a preference with two properties, that is, *direction-independence* and *invariance with a simultaneous shift of alternatives*: as we discussed, Axiom 7 guarantees the former property. By contrast, Axiom 5 (i.e., balanced indifference) demands the latter property because it preserves a strict preference and indifference provided the signed measure $p-q$ is unchanged. Accordingly, the latter axiom plays the role of the structural axiom additionally imposed by semiorder to interval order, and assuming interval order along with the EUDS axioms successfully obtains a constant JND function.[19] By contrast, the interval order only needs such a structural

[19]Semiorder assumes (a) the transitivity of strict preference $\succ$, (b) the interval order axiom

axiom, which guarantees the latter property of the preference, to derive a constant JND function, whereas it already incorporates the former.

Second and more importantly, our axiomatization is more behaviorally compelling than that in previous studies because we impose the independence axioms (i.e., irresolute independence, strict preference convexity, and balanced indifference) on the preference pair $(\succ, \sim)$, which is directly observable from choice, whereas in the previous studies, researchers imposed independence on the *transitive core*, which can only be derived from the preference.[20] As Fishburn (1968) indicated, demanding the standard independence axiom of the semiorder preference *eliminates* intransitivity, which reduces to the regular (i.e., transitive) expected utility. In previous studies, such a problem was circumvented by applying independence to the transitive core rather than to the preference. However, we address such an issue by *weakening* independence imposed on the preference rather than on the transitive core, which is the driving force to characterize a broad range of intransitive indifference.[21]

Note that the finiteness of the JND function entails that EUOS only involves intransitive indifference caused by imperfect discrimination, that is, the first class of intransitive indifference discussed in Section 4.2. It follows from Theorem 3 that the utility function u can be entirely revealed by the transitive core, which admits the *two-directional* implications of the utility function; that is, for all p, $q \in \mathcal{P}(X)$, $p \succsim^{TC} q$ if and only if $\int_X u(x)dp \geq \int_X u(x)dq$ in EUOS. The latter result also derives a stronger uniqueness property (i.e., uniqueness up to a positive affine transformation) for EUOS.

(Axiom 7), and (c) for all p, q, $r \in \mathcal{P}(X)$, $p \succ q \succ r$ implies that $s \succ r$ or $p \succ s$ for all $s \in \mathcal{P}(X)$. (This axiomatization of the semiorder is an adaptation of that by Scott and Suppes (1958).) Condition (c) guarantees that no preference interval can be included in an indifference interval (i.e., it excludes the possibility that $p \succ q \succ r$, $s \sim p$, and $s \sim r$ for some alternatives p, q, r, and s), which implies that the JND function is invariable with a change of the benchmark (or inferior) alternative and thus, establishes a constant JND function, along with some additional assumptions (e.g., Dalkiran et al., 2018; Scott and Suppes, 1958). Alternatively, the above conditions, except for (c), typically derive a non-constant JND function (e.g., Fishburn, 1970a).

[20] In our notation, the independence axiom in the previous studies states that $p \sim^{TC} q$ implies $\frac{1}{2}p + \frac{1}{2}r \sim^{TC} \frac{1}{2}q + \frac{1}{2}r$ for all p, q, $r \in \mathcal{P}(X)$.

[21] Recall that irresolute independence and strict preference convexity are reminiscent of the axioms suggested by Fishburn (1968) for analyzing intransitive indifference.

5.2 Categorical sensitivity

Next, we focus on categorical sensitivity, wherein only complete sensitivity or complete insensitivity is allowed for all alternative pairs.

First, we strengthen the irresolute independence axiom.

Axiom 3' (Independence) For all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in (0,1)$, the following statements hold:

(a) $p \succ q$ implies $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$;

(b) $p \sim q$ implies $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$.

The only difference between Axioms 3 and 3' is that we demand a strict, rather than a weak, preference (i.e., we rule out indifference) in the latter half of Axiom 3'(a); that is, once the DM has exhibited a strict preference for alternative p over q, mixing the alternatives with the third alternative r never alters the strict preference. Consequently, we establish a strict preference between any alternatives p' and q', such that the signed measure $p' - q'$ parallels $p - q$ and $p \succ q$. Note that Axiom 3' also implies Axioms 4 and 5, as we mentioned.

Now, we consider the following special case of EUDS.

Definition 7 (Expected utility with categorical sensitivity) EUDS (u, δ') is referred to as the *expected utility with categorical sensitivity (EUCS)* if either $\delta'(p,q) = 0$ or $\delta'(p,q) = +\infty$ for all $(p,q) \in H(u)$.

Hence, EUCS only assumes complete sensitivity (i.e., $\delta'(p,q) = 0$) or complete insensitivity (i.e., $\delta'(p,q) = +\infty$) for all alternative pairs (p,q). The next theorem axiomatizes this special case.

Theorem 6 *The following statements are equivalent.*

(a) Preference pair $(\succ, \sim)$ satisfies Axioms 1, 2, 3', and 6.

(b) Preference pair $(\succ, \sim)$ admits an EUCS representation (u, δ').

The proof is in the Appendix. Intuitively, independence retains a strict preference after mixing alternatives with another alternative. Thus, it excludes a nonzero and finite value of the JND function, which establishes EUCS. Because EUCS admits no alternative pair (p, q) such that $0 < \delta'(p, q) < +\infty$, it only allows for intransitive indifference caused by uncertainty about tastes, that is, the second class of intransitive indifference discussed in Section 4.2. Accordingly, the set $\mathcal{U}$ of utility functions is generally non-singleton and thus, its uniqueness result follows from Theorem 2.

For EUCS, we can also establish the following corollary of Theorem 4.

Corollary 2 *Assume that $(\succ, \sim)$ admits an EUCS representation (u, δ') and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p, q) \in H(u)$ be such that $(p, q) \notin \partial(I)$. Then, the following statements hold:*

(a) $p \succ q$ if and only if $p \succ^{TC} q$ (or equivalently, $p \succ_{\mathcal{U}}^{EMU} q$);
(b) $p \sim q$ if and only if $p \bowtie^{TC} q$ (or equivalently, $p \bowtie_{\mathcal{U}}^{EMU} q$).

Because EUCS *excludes* intransitive indifference caused by imperfect discrimination, which yields a finite value of the JND function, we can directly link a strict preference and indifference in EUCS to a strict preference and indecisiveness in the transitive core (or the corresponding EMU); that is, indifference in the former framework can be relabeled as indecisiveness in the latter for *any* alternative pair (p, q) $\in H(u)$, except for those on the boundary of I. An argument similar to Corollary 1 also establishes the link between indifference and indecisiveness on the boundary of I by focusing on $\widehat{\succ}^{TC}$ (i.e., the algebraic interior of $\succ^{TC}$) instead of $\succ^{TC}$.

In EUCS, $p \succsim q$ for some p, $q \in \mathcal{P}(X)$ whenever *at least one* utility function u exists in $\mathcal{U}$, such that alternative p yields a higher expected utility than q, that is, $\int_X u(x)dp \geq \int_X u(x)dq$. The latter property is reminiscent of the justifiability model (Lehrer and Teper, 2011), wherein a weak preference holds whenever at least one belief establishes a higher subjective expected utility for one alternative than for the other. The justifiability model is generally complete and intransitive, and, as Nishimura (2018) indicated, its transitive core becomes Bewley's (1986) Knightian uncertainty model, which is incomplete and transitive. Accordingly, Corollary 2 parallels Nishimura's result in the sense that the transitive core relates EUCS, which is a complete and intransitive model, to EMU, which is an incomplete and transitive model. However, a major difference between the two approaches is that we focus on uncertainty about *tastes*, whereas the existing approach characterizes uncertainty about *beliefs*.

6 Related literature

In this section, we review the literature relevant to this study. First, EUDS is relevant to a wide range of intransitive indifference models. Semiorder, one of the most prevailing frameworks of intransitive indifference, was initially developed by Luce (1956) for the preference over prizes and has been extended in many studies (e.g., Dalkiran et al., 2018; Gilboa and Lapson, 1995; Vincke, 1980). Interval order (Fishburn, 1970a) and strict partial order (Suppes, 1957) also generalize the semiorder approach.

Among others, Vincke (1980) and Dalkiran et al. (2018) also obtained a constant JND function in choice over *lotteries* rather than over prizes, as we do. However, as noted in Section 5.1, we characterize a constant JND function as a special case of EUDS rather than directly axiomatizing it. In such an analysis, assuming a strict partial order, which is more general than semiorder, and directly applying the independence axioms to the preference, rather than to its transitive core, play a crucial role. Next, Gilboa and Lapson (1995) characterized intransitive indifference in a product space, which can be interpreted as the set of multiattribute goods, as in our motivating examples. However, they also obtained a direction-*in*dependent JND function, as did Luce (1956), because they assumed a semiorder. Accordingly, their model cannot accommodate intransitive indifference caused by uncertainty and thus cannot typically explain the motivating examples highlighted in the Introduction.

Second, the value of the JND function can be interpreted as a degree of indifference, that is, how robust the indifference between alternatives is, which is reminiscent of preference intensity (Fishburn, 1970c; Gerasimou, 2021) and grades of indecisiveness (Minardi and Savochkin, 2015). A binary relation over *alternative pairs* was used in these previous studies on preference intensity and grades of indecisiveness;

however, we axiomatize EUDS from a binary relation over *alternatives*, which can be more easily associated with the standard choice-theoretic frameworks, such as EMU and the transitive core. Another relevant approach is to use the *decision time* (response time) as a measure of the difficulty of choice (e.g., Echenique and Saito, 2017; He and Natenzon, 2020; Koida, 2017). In particular, Koida related the decision time to the *angle* formed between the vector generated by the utility difference of alternatives and the indifference curve, which can be comparable with the direction-dependent intransitive indifference in this study. The literature on the decision time typically assumes that a transitive strict preference manifests eventually; however, we allow indifference to be persistent and intransitive.

7 Concluding remarks

In this study, we axiomatized EUDS, which is an intransitive indifference model that exhibits direction-dependent sensitivity to a utility difference of alternatives. We also specified two sources of intransitive indifference, that is, imperfect discrimination and uncertainty about tastes, by allowing both finite and infinite values of the JND function. EUDS is relevant but not equivalent to an incomplete preference model, that is, EMU, which we characterize with the transitive core. A possible future study would be to generalize EUDS to explore connections between the current approach and others, such as preference intensity, grades of indecisiveness, and decision time.

Appendix

Proof of Theorem 1 The necessity part of the theorem is straightforward.

The sufficiency part is as follows: define $\hat{D}(\succ)$, $D(\succ) \subseteq \mathrm{ca}(X)$, such that

$$\hat{D}(\succ) \equiv \{\lambda(p-q) : p, q \in \mathcal{P}(X),\ \lambda > 0,\ p \succ q\}$$

and

$$D(\succ) \equiv \mathrm{cl}(\hat{D}(\succ)); \tag{2}$$

that is, $\hat{D}(\succ)$ is the set generated by signed measures $\lambda(p-q)$, wherein $\lambda > 0$ and p is strictly preferred to q, and $D(\succ)$ is its closure in the weak*-topology. The set

$D(\succ)$ is nonempty because of nontriviality.

The following lemma characterizes $D(\succ)$.

Lemma 1 The following statements hold:
(a) $D(\succ)$ is a closed convex cone.
(b) For all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies $p - q \in D(\succ)$.
(c) For all p, $q \in \mathcal{P}(X)$, $p - q \in D(\succ)$ implies $p \succsim q$.

Proof To prove statement (a), we first note that $D(\succ)$ is a closed cone by definition. We also indicate that $D(\succ)$ is convex. Suppose that, for some p, q, p', $q' \in \mathcal{P}(X)$, $p - q$, $p' - q' \in \hat{D}(\succ)$. Without loss of generality, we assume that $p \succ q$ and $p' \succ q'$. If, say, $p \sim q$, there exist $\hat{p}$, $\hat{q} \in \mathcal{P}(X)$ and $\lambda \in [0, 1]$, such that $\hat{p} - \hat{q} = \lambda(p - q)$ and $\hat{p} \succ \hat{q}$, because $p - q \in \hat{D}(\succ)$. Then, we use $\hat{p}$ and $\hat{q}$ in place of p and q, and proceed accordingly. It follows from strict preference convexity that for all $\alpha \in [0, 1]$, $\alpha p + (1 - \alpha)p' \succ \alpha q + (1 - \alpha)q'$, which indicates that $\alpha(p - q) + (1 - \alpha)(p' - q') \in \hat{D}(\succ)$; accordingly, $\hat{D}(\succ)$ is convex, which also establishes the convexity of $D(\succ)$.

Next, statement (b) is implied by definition. To indicate statement (c), we first note that $D(\succ)$ is equivalent to the algebraic closure of $\hat{D}(\succ)$ because $\hat{D}(\succ)$ is convex and has a nonempty interior, the latter of which follows from nontriviality. Accordingly, we indicate that, for all p, $q \in \mathcal{P}(X)$, $p - q \in$ *al-cl*$(\hat{D}(\succ))$ implies that $p \succsim q$, wherein *al-cl*$(\cdot)$ denotes the algebraic closure of $(\cdot)$. Assume, contrariwise, that $p - q \in$ *al-cl*$(\hat{D}(\succ))$ and $q \succ p$ for some p, $q \in \mathcal{P}(X)$. Let p', $q' \in \mathcal{P}(X)$ be such that $p' \succ q'$ and thus, $p' - q' \in \hat{D}(\succ)$. Weak continuity implies that $A \equiv \{\alpha \in [0, 1] : \alpha q + (1 - \alpha)q' \succ \alpha p + (1 - \alpha)p'\}$ is open and nonempty. However, the latter property entails that, for all such p' and q', there exists some $\alpha \in (0, 1)$ such that $\alpha q + (1 - \alpha)q' \succ \alpha p + (1 - \alpha)p'$, which in turn implies that $\alpha(p - q) + (1 - \alpha)(p' - q') \notin \hat{D}(\succ)$ and contradicts $p - q \in$ *al-cl*$(\hat{D}(\succ))$. **Q.E.D.**

We define

$$\mathcal{U} \equiv \left\{ u \in C(X) : \int_X u(x) d\mu \geq 0 \text{ for all } \mu \in D(\succ) \right\}, \tag{3}$$

which is a nonempty closed convex cone in $C(X)$. The following lemma relates $\mathcal{U}$ to $D(\succ)$ in a manner similar to that in the paper by Dubra et al. (2004).

Lemma 2 For all p, $q \in \mathcal{P}(X)$, $\int_X u(x)dp \geq \int_X u(x)dq$ for all $u \in \mathcal{U}$ if and only if $p - q \in D(\succ)$.

Proof The "if" part holds by definition.

Consider the "only if" part. Suppose, contrariwise, that, for some p, $q \in \mathcal{P}(X)$, such that

$$\int_X u(x)dp \geq \int_X u(x)dq \tag{4}$$

for all $u \in \mathcal{U}$, the sets $\{p - q\}$ and $D(\succ)$ are disjoint. Because $D(\succ)$ is a closed convex cone under the weak*-topology, the Hahn–Banach separation theorem (Aliprantis and Border, 2006) entails that there exist a continuous linear functional L on $\mathrm{ca}(X)$ and a $\alpha \in \Re$ such that $L(\mu') \geq \alpha > L(p - q)$ for all $\mu' \in D(\succ)$.

Because $0 \in D(\succ)$, $0 = L(0) \geq \alpha > L(p - q)$. However, because $D(\succ)$ is a cone, $kL(\mu) = L(k\mu) \geq \alpha$ for all $\mu \in D(\succ)$ and $k > 0$, which implies that $L(\mu) \geq 0$ for all $\mu \in D(\succ)$. Accordingly, $L(\mu) \geq 0 > L(p - q)$ for all $\mu \in D(\succ)$. Next, the duality between $C(X)$ and $\mathrm{ca}(X)$ implies that there exists $v \in C(X)$ such that $L(\mu) = \int_X v(x)d\mu$ for all $\mu \in \mathrm{ca}(X)$. Thus, $\int_X v(x)d\mu' \geq 0 > \int_X v(x)d(p - q)$ for all $\mu' \in D(\succ)$. It follows that $v \in \mathcal{U}$ and $\int_X v(x)dp < \int_X v(x)dq$, which contradicts (4). **Q.E.D.**

Now, it follows from Lemmas 1 and 2 that for all $u \in \mathcal{U}$ and p, $q \in \mathcal{P}(X)$, $p \succ q$ implies $\int_X u(x)dp \geq \int_X u(x)dq$, and $\int_X u(x)dp \geq \int_X u(x)dq$ implies $p \succsim q$. Accordingly, any $u \in \mathcal{U}$ plays the role of the utility function of $\succ$.

For a given utility function $u \in \mathcal{U}$, the following lemma constructs a JND function.

Lemma 3 There exists a JND function δ given u.

Proof First, for all $(p, q) \in H(u)$, irresolute independence (a) implies that there exists $\alpha \in [0, 1]$ such that $\alpha p + (1 - \alpha)q \sim q$. Let $\bar{\alpha}(p, q)$ be the least upper bound of α that satisfies the latter condition; that is, for all $(p, q) \in H(u)$, $\bar{\alpha}(p, q) \equiv \sup\{\alpha \in [0, 1] : \alpha p + (1 - \alpha)q \sim q\}$.

Now, assume that for some $(\hat{p}, \hat{q}) \in H(u)$, $\hat{p} \succ \hat{q}$. It follows from weak continuity that $\bar{\alpha}(\hat{p}, \hat{q})\hat{p} + (1 - \bar{\alpha}(\hat{p}, \hat{q}))\hat{q} \sim \hat{q}$. Strict preference convexity implies that $\alpha\hat{p} + (1 - \alpha)\hat{q} \succ \hat{q}$ for all $\alpha > \bar{\alpha}(\hat{p}, \hat{q})$, whereas irresolute independence (b) implies that $\alpha\hat{p} + (1 - \alpha)\hat{q} \sim \hat{q}$ for all $\alpha \leq \bar{\alpha}(\hat{p}, \hat{q})$. Let $r_{p,q} \equiv \bar{\alpha}(p, q)p + (1 - \bar{\alpha}(p, q))q$ and $\delta(p, q) \equiv \int_X u(x)dr_{p,q} - \int_X u(x)dq$. For all p, $q \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p - q = \lambda(\hat{p} - \hat{q})$, balanced indifference entails that $r_{p,q} - q = r_{\hat{p},\hat{q}} - \hat{q}$, which obtains $\delta(p, q) = \delta(\hat{p}, \hat{q})$. Because the choice of such a $(\hat{p}, \hat{q})$ is arbitrary, the latter argument guarantees condition (a) in Definition 2.

Second, to indicate condition (b) in Definition 2, let p, p', $q \in \mathcal{P}(X)$ be such that $p \succ q$ and $p' \succ q$. It follows from strict preference convexity that $\alpha p + (1-\alpha)p' \succ q$ for all $\alpha \in [0, 1]$, which implies that $\delta(\alpha p + (1 - \alpha)p', q) \leq \alpha\delta(p, q) + (1 - \alpha)\delta(p', q)$.

Third, assume that for some $(\hat{p}, \hat{q}) \in H(u)$, $\hat{p} \sim \hat{q}$. If there exists $p' \in \mathcal{P}(X)$ such that $p' \succ \hat{q}$ and $\hat{p} = \alpha p' + (1 - \alpha)\hat{q}$ for some $\alpha \in (0, 1]$, use p' in place of $\hat{p}$ and apply the argument in the second paragraph of this proof. Furthermore, if there exists no such p', irresolute independence (b) implies that $p'' \sim \hat{q}$ for all $p'' \in \mathcal{P}(X)$ such that $\hat{p} = \alpha p'' + (1 - \alpha)\hat{q}$ for some $\alpha \in (0, 1]$. We can naturally define $\delta(\hat{p}, \hat{q}) = +\infty$ in this case, which also entails the second half of condition (c) in Definition 2.

Finally, for all p, $q \in \mathcal{P}(X)$, such that $\int_X u(x)dp \leq \int_X u(x)dq$, we can consistently define $\delta(p, q) = +\infty$, which obtains the first half of condition (c) in Definition 2. **Q.E.D.**

Proof of Theorem 2 We define $D(\succ)$ and $\mathcal{U}$ as in (2) and (3) in the proof of Theorem 1. By definition, $\mathcal{U}$ is closed and convex.

First, $u \in \langle\mathcal{U}\rangle$ implies that u is a utility function of $\succ$, by construction. Note that $u \in \langle\mathcal{U}\rangle$ if and only if $\alpha u + \beta \in \langle\mathcal{U}\rangle$ for all $\alpha > 0$ and $\beta \in \Re$. Conversely, assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and suppose, contrariwise, that $u \notin \langle\mathcal{U}\rangle$. It follows that $\int_X u(x)dp < \int_X u(x)dq$ for some p, $q \in \mathcal{P}(X)$, such that $p - q \in D(\succ)$. Without loss of generality, we assume that $p - q \in int(D(\succ))$ ($int(\cdot)$ denotes the interior of $(\cdot)$ with respect to the weak*-topology). Then, there exist p', $q' \in$

$\mathcal{P}(X)$ and $\lambda > 0$ such that $p' - q' = \lambda(p - q)$ and $p' \succ q'$. However, because $\int_X u(x)dp$ $< \int_X u(x)dq$ and $p' - q' = \lambda(p - q)$, $\int_X u(x)dp' < \int_X u(x)dq'$, which contradicts u being a utility function of $\succ$.

Finally, Lemma 3 entails that the JND function δ_u can uniquely be defined once a utility function $u \in \langle \mathcal{U} \rangle$ has been determined. **Q.E.D.**

Proof of Theorem 3 First, because $\succsim$ is complete, Definition 4 can be restated as follows, using the strict, rather than the weak, preference: for all p, $q \in \mathcal{P}(X)$,

$$p \succsim^{TC} q \text{ if and only if} \begin{cases} r \succ p \text{ implies } r \succ q \\ q \succ r \text{ implies } p \succ r \end{cases} \quad \text{for all } r \in \mathcal{P}(X).$$

Now, assuming that $p \succsim^{EMU}_{\mathcal{U}} q$ for some p, $q \in \mathcal{P}(X)$, we obtain $\int_X u(x)dp \geq \int_X u(x)dq$ because $u \in \mathcal{U}$. It follows from condition (a) in Definition 2 that $\delta(p', q')$ $= \delta(p'', q'')$ for all p', p'', q', $q'' \in \mathcal{P}(X)$ and $\lambda > 0$, such that $p'' - q'' = \lambda(p' - q')$. Accordingly, we obtain $U_p(\succ) \subseteq U_q(\succ)$ and $L_p(\succ) \supseteq L_q(\succ)$, and thus, for all $r \in \mathcal{P}(X)$, $r \succ p$ implies that $r \succ q$, whereas $q \succ r$ implies that $p \succ r$, and it follows that $p \succsim^{TC} q$.

Conversely, assume that $p \succsim^{TC} q$ and $p \not\succsim^{EMU}_{\mathcal{U}} q$. Because $\mathcal{U}$ is defined by (3), the latter condition obtains $p - q \notin D(\succ)$ for the closed convex cone $D(\succ)$ defined in (2) or equivalently, $p \notin D(\succ) + q$, which implies that $(D(\succ) + p) \setminus (D(\succ) + q) \neq \phi$. (Note that $0 \in D(\succ)$.) Without loss of generality, we assume that, for some $s \in \mathcal{P}(X)$, $s \succ p$ (i.e., $s \in D(\succ) + p$) and $s \notin D(\succ) + q$. However, the latter conditions imply that $q \succsim s \succ p$, which contradicts the assumption that $p \succsim^{TC} q$. **Q.E.D.**

Proof of Theorem 4 First, because $\mathcal{U}$ is the set defined in Theorem 2, it satisfies (3) with the set $D(\succ)$ defined by (2).

Now, we prove statement (a). If $(p, q) \in H(u)$ and $\delta(p, q) < +\infty$, then there exist p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p' \succ q'$ and $p' - q' = \lambda(p - q)$, which entail that $p - q \in int(D(\succ))$. Accordingly, $\int_X u'(x)dp \geq \int_X u'(x)dq$ for all $u' \in \mathcal{U}$ and the strict inequality holds for at least one $u' \in \mathcal{U}$; that is, we obtain $p \succ^{EMU}_{\mathcal{U}} q$. The converse implication is straightforward.

Next, to indicate statement (b), we first prove the following lemma, showing that an infinity value of the JND function for alternative pair (p, q) establishes a utility function $\hat{u} \in \mathcal{U}$ that yields an equal expected utility for p and q.

Lemma 4 For a given EUDS representation (u, δ) and all $(p, q) \in H(u)$, $\delta(p, q) = +\infty$ if and only if there exists $\hat{u} \in \mathcal{U}$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$.

Proof For a given EUDS representation (u, δ) and some $(p, q) \in H(u)$, let $\delta(p, q)$ $= +\infty$. If $\int_X u(x)dp = \int_X u(x)dq$, the desired result can trivially hold. Accordingly, we assume that $\int_X u(x)dp \neq \int_X u(x)dq$, and it follows that $\int_X u(x)dp > \int_X u(x)dq$ or $\int_X u(x)dp < \int_X u(x)dq$. Without loss of generality, we assume that $\int_X u(x)dp > \int_X u(x)dq$.

First, suppose that there exists $u' \in \mathcal{U}$ such that $\int_X u'(x)dp < \int_X u'(x)dq$. Because $\mathcal{U}$ is convex, there exists $\alpha \in [0, 1]$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$ with $\hat{u} \equiv \alpha u + (1 - \alpha)u'$, which obtains the desired result. Second, suppose that $\int_X u'(x)dp$ $> \int_X u'(x)dq$ for all $u' \in \mathcal{U}$. By the definition of $\mathcal{U}$, there exists p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$, such that $p' \succ q'$ and $p' - q' = \lambda(p - q)$. However, it follows that $\delta(p, q) < +\infty$, which is a contradiction. Finally, if there exists $u' \in \mathcal{U}$ such that $\int_X u'(x)dp = \int_X u'(x)dq$, the desired result immediately follows.

Conversely, for some $(p, q) \in H(u)$, assume that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$ for some $\hat{u} \in \mathcal{U}$. It follows from Definition 2(c) that $\delta(p, q) = +\infty$. **Q.E.D.**

Now, statement (b) trivially holds if *int*(I) is empty. Accordingly, we assume that there exists some $(p, q) \in$ *int*(I), and thus, $\delta(p, q) = +\infty$. It follows from Lemma 4 that there exists $\hat{u} \in \mathcal{U}$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$. Because (p, q) is in the interior of I, there exist $p' \in \mathcal{P}(X)$ and $\epsilon > 0$, such that $(p', q) \in I$, $p' \in N_\epsilon(p)$, and $\int_X \hat{u}(x)dp' < \int_X \hat{u}(x)dq$. However, Lemma 4 also guarantees the existence of u' $\in \mathcal{U}$ such that $\int_X u'(x)dp' = \int_X u'(x)dq$, which also obtains $\int_X u'(x)dp > \int_X u'(x)dq$ by construction. Similarly, $(p, q) \in$ *int*(I) implies that there exist $p'' \in \mathcal{P}(X)$ and $\epsilon > 0$, such that $(p'', q) \in I$, $p'' \in N_\epsilon(p)$, and $\int_X \hat{u}(x)dp'' > \int_X \hat{u}(x)dq$. Again, Lemma 4 guarantees the existence of $u'' \in \mathcal{U}$, such that $\int_X u''(x)dp'' = \int_X u''(x)dq$

and $\int_X u''(x)dp < \int_X u''(x)dq$. Accordingly, we establish $p \bowtie_{\mathcal{U}}^{\text{EMU}}$ because $\int_X u'(x)dp > \int_X u'(x)dq$ and $\int_X u''(x)dp < \int_X u''(x)dq$.

Conversely, assume that $p \bowtie_{\mathcal{U}}^{\text{EMU}}$, which implies that $\int_X u'(x)dp > \int_X u'(x)dq$ and $\int_X u''(x)dp < \int_X u''(x)dq$ for some u', $u'' \in \mathcal{U}$. Because $\mathcal{U}$ is convex, there exists $\alpha \in [0,1]$ and $\hat{u} \equiv \alpha u' + (1-\alpha)u''$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$. It follows from Lemma 4 that $\delta(p,q) = +\infty$. **Q.E.D.**

Proof of Theorem 5 The sufficiency part is straightforward.

Conversely, assume that Axioms 1–7 are satisfied. It follows from Axioms 1–6 that preference $(\succ, \sim)$ admits an EUDS representation (u, δ). Let $\mathcal{U}$ be the set defined in Theorem 2.

First, we indicate that Axiom 7 together with the EUDS preference implies that, for all p_1, p_2, $q \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, $p_1 \sim q$ and $p_2 \sim q$ entail that $p_\alpha \equiv \alpha p_1 + (1-\alpha)p_2 \sim q$ for all $\alpha \in [0,1]$. Suppose, contrariwise, that there exist p_1, $p_2 \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, such that $p_i \sim q$ for $i = 1$, 2 and $p_\alpha \succ q$. Without loss of generality, we assume that $\alpha = 1/2$ and let $p \equiv p_{\frac{1}{2}}$. Let $p' = p + (p - p_1) = p_2$, $q' = q + (p - p_1)$, $p'_1 = p_1 + (p - p_1) = p$, and $p'_2 = p_2 + (p - p_1)$; that is, p', q', and p'_i for $i =$ 1, 2 are derived from shifting the corresponding alternatives by $p - p_1$. Because the EUDS preference is invariable with a simultaneous shift of alternatives, we obtain $p' \succ q'$, $p = p'_1 \sim q'$, and $p' = p_2 \sim q$. However, it follows that $p \succ q$, $p' \succ q'$, $p \sim q'$, and $p' \sim q$. The latter statement violates Axiom 7, which is a contradiction.

Next, for all $q \in \mathcal{P}(X)$, let B_q be the intersection between the indifference curve and the closure of the strict upper contour set, given q; that is, $B_q \equiv \{\hat{p} \in \mathcal{P}(X) : \hat{p} \sim q\} \cap \text{cl}(U_q)$, which is nonempty by nontriviality. It follows from the argument in the previous paragraph that the (thick) indifference curve is a convex set in $\mathcal{P}(X)$. Because strict preference convexity also establishes the convexity of $\text{cl}(U_q)$, $p_\alpha \in B_q$ for all $\alpha \in [0,1]$ if and only if p_1, $p_2 \in B_q$.

Now, fix $u \in \mathcal{U}$ so that $\int_X u(x)dp_1 = \int_X u(x)dp_2$ for some p_1, $p_2 \in B_q$. Then, for all p_1, $p_2 \in B_q$ and $\alpha \in [0,1]$, $p_\alpha \in B_q$ and thus, $k \equiv \delta_u(p_1, q) = \delta_u(p_2, q) = \delta_u(p_\alpha, q)$. Because k is invariant with a shift of alternatives, we obtain an EUOS representation (u, k). By construction, u in the EUOS representation is unique up to a positive affine transformation. **Q.E.D.**

Proof of Theorem 6 The sufficiency part is straightforward.

Conversely, assume that Axioms 1, 2, 3', and 6 are satisfied. Because independence implies irresolute independence, strict preference convexity, and balanced indifference, preference $(\succ, \sim)$ admits an EUDS representation (u, δ').

Now, suppose, contrariwise, that $0 < \delta'(p, q) < +\infty$ for some p, $q \in \mathcal{P}(X)$. This entails that there exist p', q', p'', $q'' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p'' - q'' = \lambda(p' - q')$, $p' \sim q'$, and $p'' \succ q''$. However, it follows from Axiom 3' that

$$\frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}q'' \sim \frac{\lambda}{1+\lambda}p' + \frac{1}{1+\lambda}q'' = \frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}p'' \succ \frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}q'',$$

wherein the equality is derived from the construction, which is a contradiction. Accordingly, we obtain $\delta'(p, q) = 0$ or $\delta'(p, q) = +\infty$ for all $(p, q) \in H(u)$. **Q.E.D.**

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