Persistence of Termination of Term Rewriting Systems with Ordered Sorts

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Abstract

A property P is persistent if for any many-sorted term rewriting system $\mathcal{R}$, $\mathcal{R}$ has the property P if and only if term rewriting system $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property P. Zantema showed that termination is persistent for term rewriting systems without collapsing or duplicating rules. In this paper, we show that the Zantema's result can be extended to term rewriting systems on ordered sorts, i.e., termination is persistent for term rewriting systems on ordered sorts without collapsing, decreasing or duplicating rules. Furthermore we give the example as application of this result. Also we obtain that completeness is persistent for this class of term rewriting systems.
Keywords:term rewriting system, termination, persistence, ordered sort

1 Introduction

Term rewriting systems (TRSs) can offer both flexible computing and effective reasoning with equations and have been widely used as a model of functional and logic programming languages and as a basis of theorem provers, symbolic computation, algebraic specification and verification [3].

A rewrite system is called *terminating (strongly normalizing)* if there is no infinite rewrite sequence. The notion of termination for rewrite systems corresponds to the existence of answers of computations. So termination is the fundamental notion of term rewriting systems as computation models [6]. It is well-known that termination is undecidable for term rewriting systems in general. However, several sufficient approaches for proving this property have been successfully developed in particular cases.

Zantema [17] introduced the notion of *persistence* as follows. A property P is persistent if for any many-sorted TRS $\mathcal{R}$, $\mathcal{R}$ has the property P if and only if TRS $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property P. Usual many-sorted TRS was extended with ordered sorts by Aoto and Toyama [2]. And it was shown that the persistency of confluence [1] is preserved for this extension in [2]. Zantema [17] showed that termination is persistent for TRSs without collapsing or duplicating rules. Ohsaki and Middeldorp [14] studied the persistence of termination, acyclicity and non-loopingness on equational many-sorted TRSs.

In this paper, we show that the above Zantema's result is preserved for Aoto and Toyama's extension in the subclass of order sorted term rewriting systems.

This research was first appeared in [8] and studied in [9]. Furthermore, Ohsaki [15] studied the case of equational order-sorted TRSs. Their equational order-sorted TRSs [15] were based on ordered-sorted algebras in [7, 16]. But our TRSs on ordered sorts are based on Aoto and Toyama [2]. For example, we consider the sorts $Zero$ and Nat. If $Zero \prec Nat$ then $A_{Zero} \subset A_{Nat}$ where A_{Zero} and A_{Nat} are order-sorted algebras in equational order-sorted TRSs [15]. However, in our TRSs on ordered sorts we do not consider order-sorted algebras. In our research, if $Zero \prec Nat$ then $\mathcal{T}^{Zero} \cap \mathcal{T}^{Nat} = \emptyset$ holds where $\mathcal{T}^{Zero}$ and $\mathcal{T}^{Nat}$ are set of terms with sort $Zero$ and Nat, respectively. So our research does not depend on order-sorted algebras.

In section 2, many-sorted TRS is formulated on ordered sorts. Then, the persistence of termination on ordered sorts is shown in section 3 and 4. The proof is a modification of a simplified proof of modularity of termination [13]. Furthermore we give the example as application of this result. Also we obtain that completeness is persistent for term rewriting systems on ordered sorts.

2 Preliminaries

We mainly follow basic definitions and basic lemmas in the literature [2].

2.1 Sorted Term Rewriting Systems

In this subsection, we introduce the basic notions of sorted term rewriting systems. Usual term rewriting systems [3] are considered as special cases of sorted term rewriting systems.

Let $\mathcal{S}$ be a set of *sorts* and $\mathcal{V}$ be a set of countably infinite *sorted variables*. We assume that $\mathcal{S}$ is equipped with a well-founded partial ordering $\succ$. We write $b \succeq b'$ if and only if $b \succ b'$ or $b = b'$.

We assume there is a set $\mathcal{V}^b$ of countably infinite variables of sort b for each sort $b \in \mathcal{S}$. Let $\mathcal{F}$ be a set of sorted function symbols. We assume that each sorted function symbol $f \in \mathcal{F}$ is given with the sorts of its arguments and the sort of its value, all of which are included in $\mathcal{S}$. We write $f{:}b_1 \times \ldots \times b_n \to b'$ if and only if f takes n arguments of sorts $b_1,\ldots,b_n$ respectively to a value of sort b'. Function symbols of 0 argument are *constants*.

The set $\mathcal{T}(\mathcal{F},\mathcal{V}) = \cup_{b\in\mathcal{S}}\, \mathcal{T}(\mathcal{F},\mathcal{V})^b$ of all *sorted terms* built from $\mathcal{F}$ and $\mathcal{V}$ is defined as follows: $\mathcal{V}^b \subseteq \mathcal{T}(\mathcal{F},\mathcal{V})^b$, $f{:}b'_1 \times \ldots \times b'_n \to b'$, $t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})^{b_i}$ and $b_i \preceq b'_i$ $(i = 1,\ldots,n)$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})^{b'}$. Here $\mathcal{T}(\mathcal{F},\mathcal{V})^b$ denotes the set of all terms of sort b.

We define the set of all *strict sorted terms* if (2) is replaced by (2′) if $f{:}b'_1 \times \ldots \times b'_n \to b'$, $t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})^{b_i}$, $b_i \preceq b'_i$ $(i = 1,\ldots,n)$ and $b_j = b'_j$ whenever $t_j \in \mathcal{V}$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})^{b'}$. We write $t : b$ if t is of sort b. $\mathcal{V}(t)$ denotes the set of all variables that appear in t. $\mathcal{T}(\mathcal{F},\mathcal{V})^b$ and $\mathcal{T}(\mathcal{F},\mathcal{V})$ are abbreviated as $\mathcal{T}^b$ and $\mathcal{T}$, respectively. Let $\Box^b$ be a special constant (*hole*) of sort b. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box^b \mid b \in \mathcal{S}\},\mathcal{V})$ are called *contexts* over $\mathcal{T}(\mathcal{F},\mathcal{V})$. We write $C{:}b_1 \times \ldots \times b_n \to b'$ if and only if the sort of context C is b' and it has n holes $\Box^{b_1},\ldots,\Box^{b_n}$. If $C{:}b'_1 \times \ldots \times b'_n \to b'$ and $t_1{:}b_1, \ldots, t_n{:}b_n$ with $b_i \preceq b'_i$ $(i = 1,\ldots,n)$ then $C[t_1,\ldots,t_n]$ denotes the term obtained from C by replacing holes with $t_1,\ldots,t_n$ from left to right. A context that contains one hole is denoted by $C[\,]$. A term t is said to be a *subterm* of s if and only if $s = C[t]$ for some context C. A *substitution* θ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ such that $x \in \mathcal{V}^b$ implies $\theta(x) \in \mathcal{T}^b$. A substitution over terms is defined as homomorphic extension. $\theta(t)$ is usually written as $t\theta$. A *sorted rewrite rule*

on $\mathcal{T}$ is a pair $l \to r$ such that $l \notin \mathcal{V}$, $\mathcal{V}(r) \subseteq \mathcal{V}(l)$, sorted terms l and r are strict and if $l : b$ and $r : b'$ then $b \succeq b'$. A *sorted term rewriting system* (STRS, for short) is a pair $(\mathcal{F}, \mathcal{R})$ where $\mathcal{F}$ is a set of sorted function symbols and $\mathcal{R}$ is a set of sorted rewrite rules on $\mathcal{T}(\mathcal{F}, \mathcal{R})$. $(\mathcal{F}, \mathcal{R})$ is often abbreviated as $\mathcal{R}$ and in that case $\mathcal{F}$ is defined to be the set of function symbols that appear in $\mathcal{R}$.

Given a STRS $\mathcal{R}$, a sorted term s is *reduced* to a sorted term t ($s \to_{\mathcal{R}} t$, in symbol) if and only if $s = C[l\theta]$ and $t = C[r\theta]$ for some rewrite rule $l \to r \in \mathcal{R}$, context C and substitution θ. We call $s \to_{\mathcal{R}} t$ a *rewrite step* or *reduction* from s to t of $\mathcal{R}$. $l\theta$ is called *redex* of this rewrite step.

The transitive reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\to^*_{\mathcal{R}}$. Terms t_1 and t_2 are *joinable* if there exists some term t' such that $t_1 \to^*_{\mathcal{R}} t' \leftarrow^*_{\mathcal{R}} t_2$. A term t is *confluent* if for any terms t_1 and t_2, t_1 and t_2 are joinable whenever $t_1 \leftarrow^*_{\mathcal{R}} t \to^*_{\mathcal{R}} t_2$. A STRS $\mathcal{R}$ is confluent if every term is confluent to $\to_{\mathcal{R}}$. A term t is a *normal form* if there is no term t' such that $t \to_{\mathcal{R}} t'$. A term t is *terminating (strongly normalizing)* if there is no infinite reduction sequence starting from term t. A STRS $\mathcal{R}$ is terminating if every term is terminating to $\to_{\mathcal{R}}$. A STRS $\mathcal{R}$ is *complete* if $\mathcal{R}$ is confluent and terminating.

If $r \in \mathcal{V}$ then the rewrite rule $l \to r$ is said to be *collapsing*. If some variable has more occurrences in r than it has l then the rewrite rule $l \to r$ is said to be *duplicating*. If $l : b$, $r : b'$ and $b \succ b'$ then the rewrite rule $l \to r$ is said to be *decreasing*.

When $\mathcal{S} = \{*\}$ with an empty relation, an STRS is called a term rewriting system (TRS, for short). Given an arbitrary STRS $\mathcal{R}$, by identifying each sort with $*$, we obviously obtain a TRS $\Theta(\mathcal{R})$ - called the underlying TRS of $\mathcal{R}$.

2.2 Sorting of Term Rewriting Systems

Aoto and Toyama [1, 2] defined the notion of *sort attachment* and formulated the notion of persistence using sort attachment. We mainly follow basic definitions in [2] in this subsection.

Let $\mathcal{F}$ and $\mathcal{V}$ be sets of function symbols and variables, respectively, on a trivial set $\{*\}$ of sorts with empty relation on it. Terms built form this language are called *unsorted terms*. Let $\mathcal{S}$ be another set of sorts with well-founded partial ordering $\succ$ on it. $\mathcal{F}$ is a set of arity fixed function symbols.

A *sort attachment* τ of $\mathcal{F}$ on $\mathcal{S}$ is a family of mapping $\cup_n(\mathcal{F}_n \to \mathcal{S}^{n+1}) \cup (\mathcal{V} \to \mathcal{S})$. We can assume that there are countably infinite variables x with $\tau(x) = b$ for each $b \in \mathcal{S}$. $f \in \mathcal{F}$ with $\tau(f) = (b_1,\ldots,b_n,b')$ is written as $f{:}b_1 \times \ldots \times b_n \to b'$. The set of *$\tau$-sorted function symbols* from $\mathcal{F}$ and that of *τ-sorted variables* from $\mathcal{V}$ are denoted by $\mathcal{F}^\tau$ and $\mathcal{V}^\tau$, respectively. A term t is said to be *well-sorted* under τ with sort b (written as $\tau(t) = b$) if and only if $t \in \mathcal{T}(\mathcal{F}^\tau, \mathcal{V}^\tau)^b$. The set of well-sorted terms under τ is denoted by $\mathcal{T}^\tau$, i.e. $\mathcal{T}^\tau = \{t \in \mathcal{T} \mid t : b \text{ for some } b \in \mathcal{S}\}$. Clearly, $\mathcal{T}^\tau \subset \mathcal{T}$. A term t is said to be *strict well-sorted* under τ with sort b if and only if $t \in \mathcal{T}(\mathcal{F}^\tau, \mathcal{V}^\tau)^b$ is strict. *Well-sorted contexts* are defined by special constants $\Box^{b_1}, \ldots$. Well-sorted terms and contexts are often treated as sorted terms and contexts, respectively.

Let $(\mathcal{F}, \mathcal{R})$ be a TRS. A sort attachment τ of $\mathcal{F}$ on $\mathcal{S}$ is said to be *consistent* with $\mathcal{R}$ if and only if for any rewrite rule $l \to r \in \mathcal{R}$, l and r are strictly well-sorted under τ and $\tau(l) \succeq \tau(r)$. The *set of τ-sorted rewrite rules* of $\mathcal{R}$ is denoted by $\mathcal{R}^\tau$. Note that $\mathcal{R}^\tau$ acts on $\mathcal{T}^\tau$, i.e. well-sorted terms $s, t \in \mathcal{T}^\tau$ whenever $s \to_{\mathcal{R}^\tau} t$; and that for any $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}^\tau} t$.

Using the sort attachment, persistence can be alternatively formulated as follows. It is clear that definition of Zantema [17] and the following definition are equivalent when set of sorts with empty relation on it.

Definition 2.1 A property P is *persistent* if and only if for any TRS $(\mathcal{F}, \mathcal{R})$ and any sort attachment τ that is consistent with $\mathcal{R}$,

$$\mathcal{R}^\tau \text{ has the property P} \Leftrightarrow \mathcal{R} \text{ has the property P.}$$

We consider the persistence of termination for TRSs on ordered sorts using definition 2.1 in this paper instead of Zantema's definition. From now on, we assume that a set $\mathcal{S}$ of sorts with well-founded strict partial order $\succ$ on it and a TRS $\mathcal{R}$ are given. Then an attachment τ on $\mathcal{S}$ that is consistent with $\mathcal{R}$ is fixed.

3 Characterization of Unsorted Terms

In this section we give a characterization of unsorted terms by ordered sorts. The proofs of the following basic lemmas were given by Aoto and Toyama [2].

Definition 3.1 The *top sort* (under τ) of an unsorted term t is defined as follows:

- $top(t) = \tau(t)$ if $t \in \mathcal{V}$.
- $top(t) = b'$ if $t = f(t_1, \ldots, t_n)$ with $f : b_1 \times \ldots \times b_n \to b'$.

Definition 3.2 Let $t = C[t_1, \ldots, t_n]$ $(n \geq 0)$ be an unsorted terms with $C[, \ldots,] \neq \Box$. We write $t = C[\![t_1, \ldots, t_n]\!]$ if and only if

1. $C{:}b_1 \times \ldots \times b_n \to b'$ is a context that is well-sorted under τ.
2. $top(t_i) \not\preceq b_i$ for all $i = 1, \ldots, n$.

The $t_1,\ldots,t_n$ are said to be the *principal subterms* of t.

We denote $t = C\,\langle\langle\, t_1, \ldots, t_n \,\rangle\rangle$ if either $t = C[\![t_1, \ldots, t_n]\!]$ or $C = \Box$ and $t \in \{t_1, \ldots, t_n\}$. Multiset $S(t)$ consists of all principal subterms of $t = C[\![t_1, \ldots, t_n]\!]$.

Definition 3.3 Let t be an unsorted term. *Rank* of t is defined as follows:

- $rank(t) = 1$ if t is well-sorted term.
- $rank(t) = 1 + max\{rank(t_1), \ldots, rank(t_n)\}$ if $t = C[\![t_1, \ldots, t_n]\!]$.

Definition 3.4 Let t be an unsorted term. *Cap* of t is defined as follows:

- $cap(t) = t$ if t is well-sorted term.
- $cap(t) = C[, \ldots,]$ if $t = C[\![t_1, \ldots, t_n]\!]$.

Definition 3.5 A rewrite step $s \to_\mathcal{R} t$ is said to be *inner* (written as $s \to^i_\mathcal{R} t$) if and only if $s = C[\![s_1,\ldots,C'\,[l\theta],\ldots,s_n]\!] \to_\mathcal{R} C\,[\![s_1,\ldots,C'\,[r\theta],\ldots,s_n]\!] = t$ for some $s_1,\ldots,s_n$, $l \to r \in \mathcal{R}$, θ and C', otherwise *outer* (written as $s \to^o_\mathcal{R} t$).

Definition 3.6 A rewrite step $s \to^o_{\mathcal{R}} t$ is said to be *vanishing* if and only if $s = C[\![s_1,\ldots, \theta(x),\ldots,s_n]\!] \to^o_{\mathcal{R}} \theta(x) = t$ for some $s_1,\ldots,s_n$, C and θ such that $C'\theta = C[s_1,\ldots,\Box,\ldots,s_n]$ and $C'[x] \to x \in \mathcal{R}$.

Definition 3.7 A rewrite rule $l \to r$ in $\mathcal{R}$ is said to be *decreasing* if and only if $top(l) \succ top(r)$. A rewrite step $s \to^o_{\mathcal{R}} t$ is said to be *decreasing* if and only if $top(s) \succ top(t)$.

Definition 3.8 A rewrite step $s \to^o_{\mathcal{R}} t$ is said to be *destructive at level 1* if and only if $s \to^o t$ is either vanishing or decreasing. The rewrite step $s \to_{\mathcal{R}} t$ is said to be *destructive at level k + 1* if and only if $s = C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i_{\mathcal{R}} C[s_1, \ldots, t_j, \ldots, s_n] = t$ with $s_j \to_{\mathcal{R}} t_j$ *destructive at level k* .

Lemma 3.9 *The following statements hold.*

1. *If a rewrite step $s \to^o_{\mathcal{R}} t$ is not vanishing then $top(s) \succeq top(t)$.*
2. *If a rewrite step $s \to^o_{\mathcal{R}} t$ is not destructive at level 1 then $top(s) = top(t)$.*

Lemma 3.10 *The following statements hold.*

1. *If $s \to_{\mathcal{R}} t$ then $rank(s) \geq rank(t)$.*
2. *If a rewrite step $s \to_{\mathcal{R}} t$ is vanishing then $rank(s) > rank(t)$.*

Definition 3.11 The *grade* ($grade(t) \in N \times \mathcal{S}$) of a term t is defined by $grade(t) = \langle rank(t), top(t)\rangle$ where N is the set of all natural numbers.

Let $>$ be the lexicographic ordering on $N \times \mathcal{S}$ induced from $>$ on N and $\succ$ on $\mathcal{S}$. The lexicographic ordering $>$ on $N \times \mathcal{S}$ is well-founded since orderings $>$ on N and $\succ$ on $\mathcal{S}$ are well-founded.

Lemma 3.12 *The following statements hold.*

1. *If $s \to_{\mathcal{R}} t$ then $grade(s) \geq grade(t)$.*
2. *If a rewrite step $s \to_{\mathcal{R}} t$ is destructive at level 1 then $grade(s) > grade(t)$.*

4 Persistence of Termination

In this section we discuss the persistence of termination for TRSs on ordered sorts and give the example as application of this result. Furthermore we obtain that completeness is persistent for TRS on ordered sorts. We mainly follow the Ohlebusch's argument in [13].

Lemma 4.1 *Let t be a strictly well-sorted term. Suppose $t = C[x_1, \ldots, x_n]$ with all variables displayed and $C : b_1 \times \ldots \times b_n \to b'$. Then for any substitution θ, if $x_i = x_j$ and $\theta(x_i)$ is a principal subterm of $t\theta$ then $\theta(x_j)$ is also a principal subterm of $t\theta$.*

Let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ be terms. We write $\langle\, s_1, \ldots, s_n \,\rangle \propto \langle\, t_1, \ldots, t_n \,\rangle$ if and only if for any $1 \leq i, j \leq n$, $s_i = s_j$ implies $t_i = t_j$.

Lemma 4.2 *If $C[\![s_1, \ldots, s_n]\!] \to^o_{\mathcal{R}} C'\langle\, \langle\, s_{i_1}, \ldots, s_{i_m} \,\rangle\, \rangle$ and $\langle\, s_1, \ldots, s_n \,\rangle \propto \langle\, t_1, \ldots, t_n \,\rangle$ then $C[t_1, \ldots, t_n] \to^o_{\mathcal{R}} C'[t_{i_1}, \ldots, t_{i_m}]$.*

Lemma 4.3 *If $s \to^o_{\mathcal{R}} t$ is a non-destructive rewrite step and applied rewrite rule is not duplicating, the multiset inclusion $S(t) \subseteq S(s)$ holds.*

Lemma 4.4

1. *If $s = C[\![s_1,\ldots,s_j,\ldots,s_n]\!] \to^i_{\mathcal{R}} C[s_1,\ldots,t_j,\ldots,s_n] = t$ is destructive at level 2 and $cap(s) \neq cap(t)$ then $S(t) = S(s) \setminus \{s_j\} \cup S(t_j)$.*
2. *If $s = C[\![s_1,\ldots,s_j,\ldots,s_n]\!] \to^i_{\mathcal{R}} C[s_1,\ldots,t_j,\ldots,s_n] = t$ is destructive at level 2 and $cap(s) = cap(t)$ then $S(t) = S(s) \setminus \{s_j\} \cup \{t_j\}$.*

Lemma 4.5 *The following statements hold.*

1. *If $s \to^o_{\mathcal{R}} t$ is not destructive at level 1 then $cap(s) \to_{\mathcal{R}^\tau} cap(t)$.*
2. *If $s \to^i_{\mathcal{R}} t$ is not destructive at level 2 then $cap(s) = cap(t)$.*

Proof.

1. Let $s = C[\![s_1,\ldots,s_n]\!] \to^o_{\mathcal{R}} C'\langle\langle s_{i_1},\ldots,s_{i_m}\rangle\rangle = t$. Since $\langle s_1,\ldots,s_n\rangle \propto \langle \Box,\ldots,\Box\rangle$ and lemma 4.2, $cap(s) = C[,\ldots,] \to_{\mathcal{R}^\tau} C'[,\ldots,] = cap(t)$.
2. Let $s = C[\![\, s_1,\ldots,s_j,\ldots,s_n]\!] \to^i_{\mathcal{R}} C[\![\, s_1,\ldots,t_j,\ldots,s_n]\!] = t$ and $s_j \to_{\mathcal{R}} t_j$. Since $cap(s)$ $= C[,\ldots,]$ and $cap(t) = C[,\ldots,]$, it is trivial. □

Definition 4.6 For any infinite rewrite sequence $D : s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \ldots$, we define the *grade* of D to be $grade(D) = grade(s_0)$.

Definition 4.7 Let $>$ is the lexicographic ordering on $N \times \mathcal{S}$ and $>^{mult}$ is the multiset extension of it. We define $\sharp s = \{grade(t) \mid t \in S(s)\}$, i.e., $\sharp s$ denote the multiset of the grades of the principal subterms of s.

Multiset ordering $>^{mult}$ is well-founded since lexicographic ordering $>$ on $N \times \mathcal{S}$ is well-founded [4].

Lemma 4.8 *If $s \to^i_{\mathcal{R}} t$ is destructive at level 2 then $\sharp s >^{mult} \sharp t$.*

Proof. Let s and t be $C[\![s_1,\ldots,s_j,\ldots,s_n]\!]$ and $C[s_1,\ldots,t_j,\ldots,s_n]$, respectively.

1. $cap(s) \neq cap(t)$. By lemma 4.4, $S(t) = S(s) \setminus \{s_j\} \cup S(t_j)$.
   - $s_j \to_{\mathcal{R}} t_j$ is vanishing. Since $rank(s_j) > rank(t_j)$ and $rank(t_j) > rank(u)$ for any $u \in S(t_j)$, $rank(s_j) > rank(u)$ holds. So $grade(s_j) > grade(u)$ for any $u \in S(t_j)$ holds. Hence, $\sharp s >^{mult} \sharp t$ holds.
   - $s_j \to_{\mathcal{R}} t_j$ is decreasing. Since $rank(s_j) \geq rank(t_j)$ and $rank(t_j) > rank(u)$ for any $u \in S(t_j)$, $rank(s_j) > rank(u)$ holds. So $grade(s_j) > grade(u)$ for any $u \in S(t_j)$ holds. Hence, $\sharp s >^{mult} \sharp t$ holds.
2. $cap(s) = cap(t)$. By lemma 4.4, $S(t) = S(s) \setminus \{s_j\} \cup \{t_j\}$.
   - $s_j \to t_j$ is vanishing. Since $rank(s_j) > rank(t_j)$, $grade(s_j) > grade(t_j)$ holds.
   - $s_j \to t_j$ is decreasing. Since $rank(s_j) \geq rank(t_j)$ and $top(s_j) \succ top(t_j)$, $grade(s_j)$ $> grade(t_j)$ holds.

Therefore, $\sharp s >^{mult} \sharp t$ holds. □

Lemma 4.9 *Let $\mathcal{R}^\tau$ be a terminating STRS. Let $D : s_0 \to_\mathcal{R} s_1 \to_\mathcal{R} s_2 \to_\mathcal{R} \ldots$ be an infinite rewrite sequence of minimal grade with respect to $\mathcal{R}$. Then the following statements hold.*

1. *There are infinitely many outer rewrite steps in D.*
2. *There are infinitely many inner rewrite steps in D which are destructive at level 2.*
3. *There are infinitely many duplicating outer rewrite steps in D.*

Proof. Since $grade(s_j) = grade(D)$ for any $j \in N$, there is no rewrite step which is destructive at level 1.

1. Suppose that there are only finitely many outer rewrite steps in D. Then we can assume that there is no outer rewrite step in D. If $s_0 = C[\![t_1, \ldots, t_n]\!]$ then there must be an infinite rewrite sequence starting from some $t_i \in S(s_0)$. Since $rank(t_i) < rank(s_0)$ this contradicts the minimality of $grade(D)$.
2. Suppose that there are only finitely many inner rewrite steps in D which are destructive at level 2. Then we can assume that there is no inner rewrite step in D which is destructive at level 2. By lemma 4.5, if $s \to^o_\mathcal{R} t$ in D then $cap(s) \to_{\mathcal{R}^\tau} cap(t)$ and if $s \to^i_\mathcal{R} t$ in D then $cap(s) = cap(t)$. By the case 1, R^τ is not terminating. This is contradiction by the assumption.
3. Suppose that there are only finitely many outer rewrite steps in which are applied duplicating rules in D. We consider the following cases.
   - If $s_j \to^o_\mathcal{R} s_{j+1}$ then $S(s_{j+1}) \subseteq S(s_j)$ since the rewrite step is non-destructive and non-duplicating and lemma 4.3. Then $\sharp s_j \geq^{mult} \sharp s_{j+1}$ holds.
   - If $s_j \to^i_\mathcal{R} s_{j+1}$ is not destructive at level 2 then we have $s_j = C[\![t_1, \ldots, t_k, \ldots, t_n]\!] \to^i_\mathcal{R} C[\![t_1, \ldots, t'_k, \ldots, t_n]\!] = s_{j+1}$ where $t_k \to_\mathcal{R} t'_k$. Then $\sharp s_j \geq^{mult} \sharp s_{j+1}$ holds since $grade(t_k) \geq grade(t'_k)$.
   - If $s_j \to^i_\mathcal{R} s_{j+1}$ is destructive at level 2 then we have $s_j = C[\![t_1, \ldots, t_k, \ldots, t_n]\!] \to^i_\mathcal{R} C[t_1, \ldots, t'_k, \ldots, t_n] = s_{j+1}$ where $t_k \to_\mathcal{R} t'_k$ is destructive at level 1. By lemma 4.8, $\sharp s_j >^{mult} \sharp s_{j+1}$. By the well-foundedness of $>^{mult}$, there are only finitely many inner rewrite steps which are destructive at level 2 in D. This contradicts the case 2. □

Lemma 4.10 *Let $\mathcal{R}^\tau$ be a terminating STRS. Assume that $\mathcal{R}$ is not terminating. Then $\mathcal{R}$ has duplicating rules and $\mathcal{R}$ has collapsing or decreasing rules.*

Proof. By lemma 4.9, it is trivial. □

Theorem 4.11 *The following statements hold.*

1. *Termination is a persistent property of TRSs on ordered sorts without collapsing and decreasing rules.*
2. *Termination is a persistent property of TRSs on ordered sorts without duplicating rules.*

Proof. By lemma 4.10, it is trivial. □

Example 4.12 We show that the following TRS $\mathcal{R}$ is terminating using theorem 4.11. To show the termination of the following TRS directly seems difficult form known results (E.g. recursive path ordering [6]). Also, we can *not* use the modularity results for composable TRSs [12] and hierarchical combinations of TRSs [11]. Furthermore, we can *not* use the Zantema's result [17] for proving termination of the following TRS. However, we can show the termination of next TRS using our results in this paper.

$$\mathcal{R} = \begin{cases} g(x,B) \to g(x,A) & (r1) \\ g(x,B) \to d(x,A) & (r2) \\ g(x,d(z,B)) \to x & (r3) \\ I(A,g(x,d(y,C))) \to I(B,g(x,d(y,C))) & (r4) \\ I(x,g(x,d(z,z))) \to d(z,z) & (r5) \\ d(z,A) \to e(z,C) & (r6) \end{cases}$$

Let $\mathcal{S} = \{0,1,2\}$, $1 \succ 0$ and $2 \succ 0$.

$$\tau = \begin{cases} g : 0 \times 0 \to 1 \\ I : 0 \times 1 \to 2 \\ d : 0 \times 0 \to 0 \\ e : 0 \times 0 \to 0 \\ A : 0, B : 0, C : 0 \end{cases}$$

Any well-sorted term in $\mathcal{T}^0$, $\mathcal{T}^1$ and $\mathcal{T}^2$ is terminating, i.e. any well-sorted term in $\mathcal{T}^\tau$ is terminating. We consider the following cases:

- $t \in \mathcal{T}^0$. Then $(r6)$ is the only applicable rule. A TRS $\{(r6)\}$ is terminating using recursive path ordering. Hence, t is terminating.
- $t \in \mathcal{T}^1$. Then $(r1)$, $(r2)$, $(r3)$ and $(r6)$ are the only applicable rules. A TRS $\{(r1),(r2),(r3),(r6)\}$ is terminating using recursive path ordering. Hence, t is terminating.
- $t \in \mathcal{T}^2$. Then $(r1)$, $(r2)$, $(r3)$, $(r4)$, $(r5)$ and $(r6)$ are the applicable rules. For any proper subterm s of t, $top(s) = 0$ or $top(s) = 1$. Since the above two cases, s is terminating. Since $top(t) = 2$, $(r4)$ and $(r5)$ are the only applicable rules to root position of term t. Hence, t is terminating.

Then STRS $\mathcal{R}^\tau$ is terminating. Since $\mathcal{R}^\tau$ has no duplicating rules and theorem 4.11, $\mathcal{R}$ is terminating.

Furthermore we obtain the persistence of completeness for TRSs on ordered sorts. The following theorem was given by Aoto and Toyama [2].

Theorem 4.13 *([2]) Confluence is a persistent property of TRSs on ordered sorts.*

Since a complete TRS is confluent and terminating, we obtain the following corollary form theorem 4.11 and theorem 4.13.

Corollary 4.14 *The following statements hold.*

1. *Completeness is a persistent property of TRSs on ordered sorts without collapsing and decreasing rules.*
2. *Completeness is a persistent property of TRSs on ordered sorts without duplicating rules.*

5 Conclusion and Future Work

In this paper, we have shown that the Zantema's result [17] is preserved for Aoto and Toyama's extension [2] in the subclass of order sorted term rewriting systems. That is, we have shown that termination is persistent for TRSs on ordered sorts without collapsing, decreasing or duplicating rules. Furthermore, we have given the example as application of our results. Also we obtain the persistence of completeness for TRSs on ordered sorts.

However, the difference between our work and Ohsaki's work [15] is still unclear. So we consider this point for the future.

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