解析学:積分の例

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1　積分の例

関数 $f(x) = \sqrt{5-x^2}$ の積分 (原始関数) を求めよ。

1.1　置換積分法を用いて積分する

定理：
$x = \phi(t)$ が C^1 級のとき、

$$\int f(x)dx = \int f(\phi(t))(\phi(t))'dt$$

1.2　変数交換

関数 $f(x) = \sqrt{5-x^2}$ は $f(\phi(t))(\phi(t))'dt$ に置換できます。
$\frac{x}{\sqrt{5}} = sin(t)$　すると、$x = \sqrt{5}sin(t)$
$f(x) = \sqrt{5}\sqrt{1-(\frac{x}{\sqrt{5}})^2}$
$f(t) = \sqrt{5}\sqrt{1-(\sin(t))^2} = \sqrt{5}\cos(t)$
$dx = \sqrt{5}\frac{d(\sin(t)}{dt}dt = \sqrt{5}\cos(t)dt$
ゆえに、
$\int f(x)dx = \int \sqrt{5}\cos(t)\sqrt{5}\cos(t)dt$
$\int f(x)dx = 5\int(\cos(t))^2dt$

また、
$(\cos(t))^2 = \frac{1}{2}(\cos(2t)+1)$

$\int f(x)dx = 5\int(\cos(t))^2dt = 5\int\frac{1}{2}(\cos(2t)+1)dt$

$\int f(x)dx = \frac{5}{2}(\int \cos(2t)dt + \int dt)$
$\int f(x)dx = \frac{5}{2}(\frac{\sin(2t)}{2} + t + c)$
$\int f(x)dx = \frac{5}{4}\sin(2t) + \frac{5}{2}t + c)$

$\frac{x}{\sqrt{5}} = sin(t)$ であるから、$t = \arcsin(\frac{x}{\sqrt{5}})$

$\sin(2t) = 2\cos(t)\sin(t) = 2\sqrt{1-(\sin(t))^2}\sin(t)$
$\sin(2t) = 2\sqrt{1-(\frac{x}{\sqrt{5}})^2}\frac{x}{\sqrt{5}} = \frac{2}{5}\sqrt{5-x^2}x$

$\int f(x)dx = \frac{5}{2}\arcsin(\frac{x}{\sqrt{5}}) + \frac{5}{4}\frac{2}{5}\sqrt{5-x^2}x + c$
$\int f(x)dx = \frac{5}{2}\arcsin(\frac{x}{\sqrt{5}}) + \frac{1}{2}\sqrt{5-x^2}x + c$

解：
$x = \sqrt{5}\sin(t)$ のとき、

$$\int f(x)dx = \int f(\phi(t))(\phi(t))'dt = \frac{1}{2}x\sqrt{5-x^2} + \frac{5}{2}\arcsin(\frac{x}{\sqrt{5}}) + c$$