定積分(3)

広義積分

定義:

閉区間(a,b)で連続な関数 $f(x)$ の一つの原始関数を $F(x)$ とすると

$$\int_{a+\varepsilon}^{b-\varepsilon} f(x)dx = \int_{a+\varepsilon}^{c} f(x)dx + \int_{c}^{b-\varepsilon} f(x)dx$$

である.

$$\int_{a}^{b} f(x)dx = \lim_{\alpha -> a} \int_{\alpha}^{c} f(x)dx + \lim_{\beta -> b} \int_{c}^{\beta} f(x)dx$$

広義積分の計算例(1)

$$\int_1^2 \frac{1}{\sqrt{2-x}}\,dx$$

$$\int_1^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{\beta\to 2}\int_1^{\beta} \frac{1}{\sqrt{2-x}}\,dx = \lim_{\beta\to 2}\left[-2\sqrt{2-x}\right]_1^{\beta}$$

$$\int_1^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{\beta\to 2}\left[-2\sqrt{2-\beta}+2\right] = 2$$

広義積分の計算例(2)

$$\int_{-\infty}^{-1} \frac{1}{x^2} dx$$

$$\int_{-\infty}^{-1} \frac{1}{x^2} dx = \lim_{\alpha \to -\infty} \int_{\alpha}^{-1} \frac{1}{x^2} dx = \lim_{\alpha \to -\infty} \left[-\frac{1}{x} \right]_{\alpha}^{-1}$$

$$\int_{-\infty}^{-1} \frac{1}{x^2} dx = \lim_{\alpha \to -\infty} \left[1 + \frac{1}{\alpha} \right] = 1$$

広義積分の計算例(3)

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$\lim_{\alpha\to -1}\int_{\alpha}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{\beta\to 1}\int_{0}^{\beta} \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$\lim_{\alpha\to -1}\left[-\sin^{-1}\alpha\right] + \lim_{\beta\to 1}\left[\sin^{-1}\beta\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

広義積分

$$a > 0$$

$$\int_a^\infty x^k dx = \lim_{\beta\to\infty}\int_a^\beta x^k dx = \lim_{\beta\to\infty}\left[\frac{x^{k+1}}{k+1}\right]_a^\beta$$

$$\int_a^\infty x^k dx = \begin{cases} -\dfrac{a^{k+1}}{k+1} & \text{収束} \quad k < -1 \\ \text{発散} \quad k \geq -1 \end{cases}$$

広義積分の計算例(4)

$$r > 0$$

$$\int_{-\infty}^{0} e^{rx} dx = \lim_{\alpha \to -\infty} \int_{\alpha}^{0} e^{rx} dx = \lim_{\alpha \to -\infty} \left[\frac{e^{rx}}{r} \right]_{\alpha}^{0}$$

$$\int_{-\infty}^{0} e^{rx} dx = \lim_{\alpha \to -\infty} \left[\frac{1 - e^{r\alpha}}{r} \right] = \frac{1}{r}$$

広義積分の計算例(5)

$$\int_0^{\infty} e^{-x}\sin x dx = \lim_{\beta\to\infty}\int_0^{\beta} e^{-x}\sin x dx$$

部分積分法を用いて計算する

$$\int_0^{\infty} e^{-x}\sin x dx = \lim_{\beta\to\infty}\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^{\beta} = \frac{1}{2}$$